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EXAMPLES

IN

Alternating-Currents

VOL. I.

SECOND EDITION
with additions

FOR

Students and Engineers

BY

rank menz

PROF. F. E. AUSTIN, B. S., E. E.

Author of "*How To Make a Transformer for Low Pressures*";
"*How To Design, Make and Operate High-Pressure Transformers*"; "*Examples in Magnetism*".

Member American Association for the Advancement of Science;
of the Society for the Promotion of Engineering Education.

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PREFACE.

One object of this book is to make money; presumably for the author. Another object of the book is to assist students who are pursuing courses in Electrical Engineering, to apply *fundamental principles* in engineering practice; of possible benefit to the reader.

While the book has been designed to assist the college student, it will be found very valuable to those who are pursuing Correspondence Courses or who are attempting the study of "*Alternating-Currents*" by themselves.

It has been stated by numerous writers and teachers that a student cannot obtain an adequate understanding of the subject of "Alternating-Currents" without performing a certain amount of numerical computation, of the nature of "*problems*". The writer has found, as a result of an experience extending over a period of many years, which has brought him into personal acquaintance with students of engineering from all over the world, that the average student requires a certain amount of *guidance*, in the solution of engineering problems that are presented to him. It is the design of this book to furnish such guidance, and at the same time lessen the labor of the "*teacher*", who will not find, however, that all of his "prerogatives" have been usurped.

No apology should ever be offered for introducing the methods of Calculus in the solution of engineering problems. If the methods of Calculus do not in all cases unfold the "whole truth", they produce results as nearly in accord with practice as are necessary in most cases; while the methods certainly insure a great saving of time in many solutions.

In very many cases no other method at present known can possibly enable one to obtain results with a desirable degree of accuracy.

If the practical applications of the methods of Calculus were properly comprehended by the student, the study of the subject would be vastly more interesting and would prove a most valuable acquisition to be used in his professional career. The *application* of some of the elementary principles of Calculus is explained in many of the "*Examples*", and in some cases the correctness of the method may be "checked" by applying some other method. The student should always seek to find a method, and apply it, to check results. There is always more than one way to *kill a problem*.

One often hears the remark from the practical man, "it may be so in theory, but it does not work that way in practice". On the other hand, he mathematician makes certain initial assumptions and after very careful athematical processes produces certain results—on paper. It is true ese theoretical results do not always coincide with the results of practice.

Why? Usually because only *part* of the theory has been incorporated into the mathematical processes. The results of practice involve *all* the theory. The human mind cannot *always* have anticipatory comprehension of the total theoretical probabilities that may constitute successful achievement.

As a matter of fact there is absolutely no *inherent* difference between theory and practice. The practitioner should not sneer at the results obtained by the theorist, nor should the theoretical mathematician hold in light esteem, the products of the practical individual. Both are indispensible as important factors of scientific progress.

However, as regards a "*problem*" that is presented for solution; given certain numerical values, and *stated fixed conditions*, definite results are properly obtained consistent with the given numerical values and conditions. Such results may or may not accord with practice; coincidence is immaterial so far as mathematical training is concerned. The solution of problems constitutes the labor of the theoretical individual, whose successful development depends upon systematic and logical processes. The professional engineer develops by solving problems—on paper, by combining the results of one problem with the results of other problems to form a well-ordered composite complete result; the whole truth; the *result found in practice*. When the complete truth is arrived at, theory and practice coincide.

If by any possibility the reader is unable to fully comprehend the mathematical methods adopted in the solution of the problems here presented, he will be able to use the *results* obtained to considerable advantage in ordinary practice. In this respect the book should prove of considerable value to the *practicing engineer*; whether mechanical, civil, or electrical.

One feature of the book not to be overlooked is the insertion from page 208 to page 216, of various "tables" containing values of variable quantities met with in engineering practice, so arranged as to include a wide range of values, and to render evaluation convenient and rapid. Special attention is directed to the tabulated values of $2\pi f$, for frequencies from 1 to 151 cycles, with corresponding values of $\frac{1}{2\pi f}$, $(2\pi f)^2$, and $\left(\frac{1}{2\pi f}\right)^2$ found on page 208. Such an arrangement of values saves a great amount of time in solving problems in classwork and in practice.

On page 213 will be found a tabular arrangement of values of capacities in *farads*, necessary to neutralize inductances from 1 to 51 henrys, and produce *resonance* in circuits, when the frequency of the applied alternating-pressure is 15, 25, 60, and 100 cycles.

A similar table may be found on page 214 for values from 0.1 to 0.99 henry.

A tabular arrangement of the reciprocals of capacities from 0.11 to 18 microfarads appears on page 211.

Other tables appear on pages [illegible] and [illegible].

Civil Engineers will be interested in the definitions on page [illegible]0.

Preface to Second Edition.

The popular demand for this book has rendered it necessary to issue a second edition very close upon the publication of the first. Notwithstanding the early production of a new edition many valuable features have been incorporated in the form of new tables and subject matter.

This book, while not designed for a text-book, has been used as such with marked success.

The author desires to express his sincere thanks to all those students, teachers, and engineers who have so kindly tendered their appreciation of this small book.

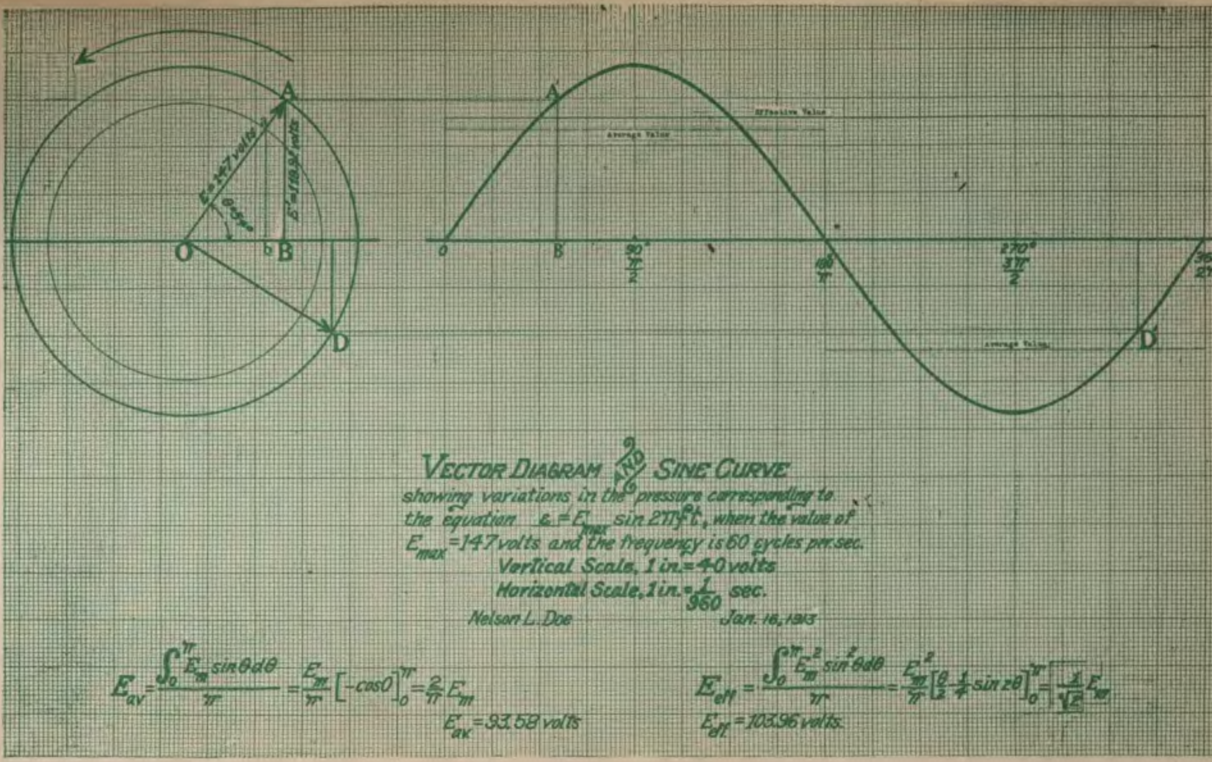
A
O
b
B
D
$E=147$ volts
$\theta=54°$
$E'=118.94$ volts
Effective Value
Average Value
A'
0
B'
90° $\frac{\pi}{2}$
180° π
270° $\frac{3\pi}{2}$
360° 2π
Average Value
D'
VECTOR DIAGRAM AND SINE CURVE
showing variations in the pressure corresponding to
the equation $e=E_{max}\sin 2\pi f t$, when the value of
$E_{max}=147$ volts and the frequency is 60 cycles per sec.
Vertical Scale, 1 in. = 40 volts
Horizontal Scale, 1 in. = $\frac{1}{960}$ sec.
Nelson L. Doe
Jan. 16, 1913
$E_{av}=\frac{\int_0^{\pi}E_m\sin\theta d\theta}{\pi}=\frac{E_m}{\pi}\left[-\cos\theta\right]_0^{\pi}=\frac{2}{\pi}E_m$
$E_{av}'=93.58$ volts
$E_{eff}=\sqrt{\frac{\int_0^{\pi}E_m^2\sin^2\theta d\theta}{\pi}}=\sqrt{\frac{E_m^2}{\pi}\left[\frac{\theta}{2}-\frac{1}{4}\sin 2\theta\right]_0^{\pi}}=\frac{1}{\sqrt{2}}E_m$
$E_{eff}'=103.96$ volts.

PLATE I.

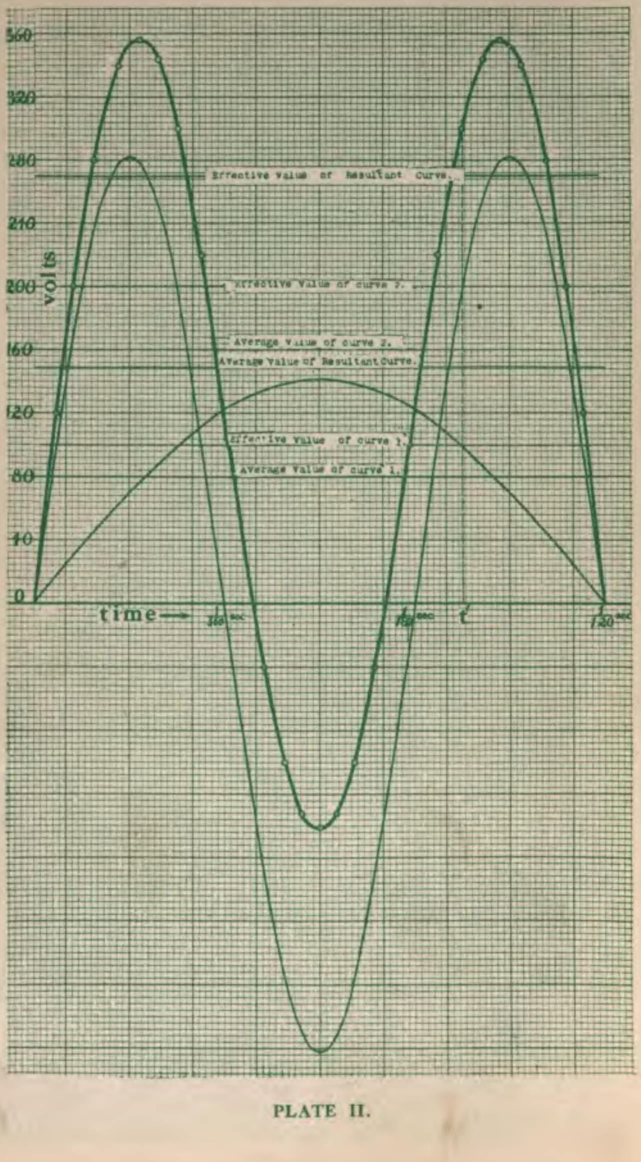
360
320
280
240
200
160
120
80
40
0
volts
time →
1/360 sec
1/180 sec
t
1/120 sec
Effective Value of Resultant Curve.
Effective Value of curve 2.
Average Value of curve 2.
Average Value of Resultant Curve.
Effective Value of curve 1.
Average Value of curve 1.

PLATE II.

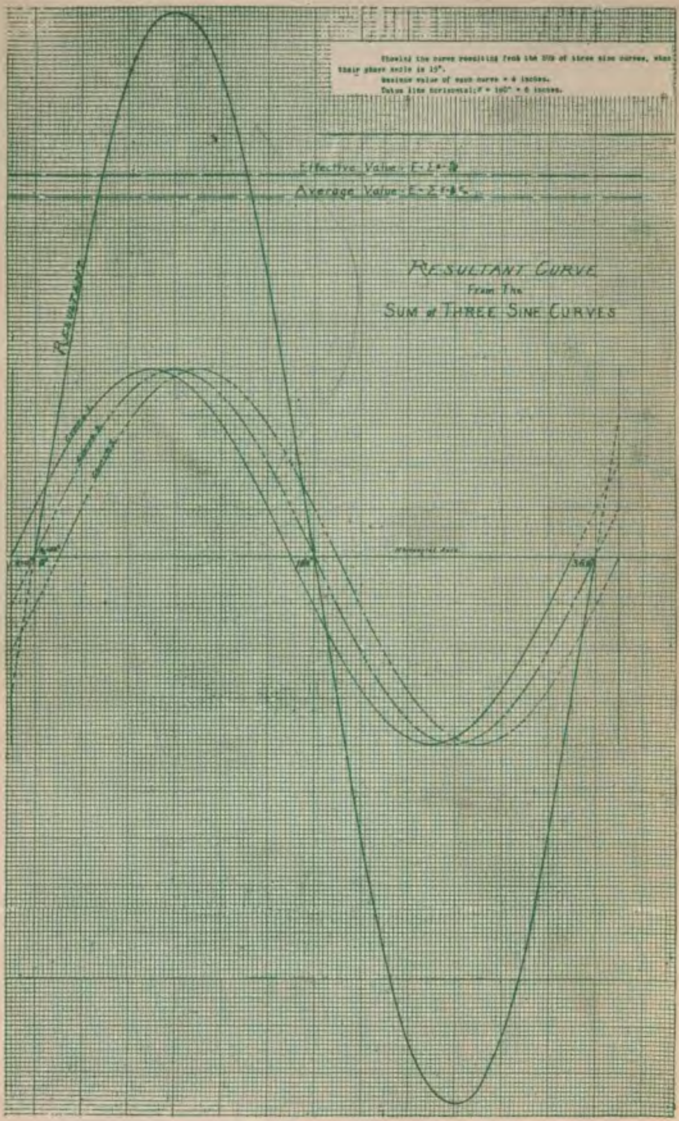
Showing the curve resulting from the sum of three sine curves, when their phase angle is 15°.
Maximum value of each curve = 4 inches.
Datum line horizontal; π = 180° = 6 inches.
Effective Value: E = Σ
Average Value: E = Σ
RESULTANT CURVE
From The
SUM of THREE SINE CURVES
RESULTANT
Horizontal Axis
180°
360

PLATE IV.

NOTATION.

The Greek letters: α, β, γ, θ, and ϕ, are employed to denote angles, in degrees. (See page 7 for Greek alphabet.) θ is usually employed to denote a *constant* angle; while ϕ usually denotes a *variable* angle.

π denotes the *ratio* of the length of the circumference of *any* circle to the length of its diameter. The length of the circumference and the length of its diameter must be expressed in the *same* units. π also denotes the unit of angular measure, called a *radian*; a unit equal to $\frac{180^\circ}{3.14159}=57^\circ\ 17'$ or $(57^\circ.295)$.

ϵ (Epsilon) denotes the *base* of the *Natural System of Logarithms.* This system is sometimes incorrectly called the "*Napierian*", and also the "*Hyperbolic*" system.

ω denotes angular velocity in *radians per second.*

$\omega=2\pi f$ (if f denotes the number of revolutions per second.)

e denotes an instaneous value of an alternating-pressure.

i denotes an instaneous value of an alternating-current.

ϕ denotes an instaneous value of an alternating-magnetic field.

E_{max} denotes a maximum value of an alternating-pressure.

I_{max} denotes a maximum value of an alternating-current.

Φ_{max} denotes a maximum value of an alternating-magnetic field.

E_{eff} denotes an effective value of an alternating-pressure.

I_{eff} denotes an effective value of an alternating-current.

Φ_{eff} denotes an effective value of an alternating-magnetic field.

E_M denotes the *effective* counter E. M. F. of a synchronous motor in volts.

E denotes any applied pressure; in volts.

E_G denotes the *effective* E. M. F. in volts of an alternator;. (Alternating-current generator.)

i denotes *any current* in amperes.

Z denotes *impedance*; expressed by $\sqrt{R^2+\left(2\pi fL-\frac{1}{2\pi fC}\right)^2}$.

g denotes *conductance.*

b denotes *susceptance.*

Y denotes *admittance.*

R denotes *ohmic resistance*, in ohms.

K or κ denotes *any constant*, or coefficient.

μ denotes magnetic *permeability.*

ρ denotes magnetic *reluctivity.* $\rho=\frac{1}{\mu}$. ρ also denotes specific resistance.

Σ denotes the *sum of*; or the summation.

$\int$ denotes the sign of integration; an old fashioned s; denoting summation of elementary areas to form a desired area.

f denotes *frequency* of any alternating quantity, *per second*.

T denotes *torque*. Also the periodic time. $\mathbf{T} = \frac{1}{f}$.

X denotes *reactance*; expressed by $\left(2\pi f \mathrm{L} - \frac{1}{2\pi f \mathrm{C}}\right)$.

x denotes a distance; usually horizontal; sometimes a *varying* length; sometimes an angle.

y denotes a distance; usually a vertical one.

d denotes a distance; usually a constant one.

l denotes a length. Also the *absolute* value of inductance coefficient.

L denotes the *coefficient* of *self-inductance*, in henrys.

M denotes the *coefficient* of mutual inductance, in henrys.

t denotes time, measured in *seconds*.

V denotes a speed of rotation, in *revolutions per minute*.

$\frac{\mathrm{V}}{60}$ denotes a speed of rotation, in *revolutions per second*.

v denotes a velocity, usually a linear velocity.

$\mathfrak{B}$ denotes intensity of magnetic field in *gausses*. (Maxwells per square centimeter.)

Φ denotes magnetic field in maxwells.

A denotes an area; either in square inches or in square cms.

$\mathfrak{R}$ denotes magnetic reluctance. $\mathfrak{R} = \frac{\Phi}{\mathrm{A}}$.

Fundamental Units.

Unit of magnetic flux, the maxwell $= \frac{1}{10}$ the C. G. S. unit.
Unit of electrical quantity, the coulomb $= \frac{1}{10}$ the C. G. S. unit.
Unit of electrical current, the ampere $= \frac{1}{10}$ the C. G. S. unit.
Unit of electrical pressure, the volt $= 10^8$ times the C. G. S. unit.

Derived Units.

Unit of resistance, the ohm, $= 10^9$ times the C. G. S. unit:
Unit of inductance, the henry, $= 10^9$ times the C. G. S. unit
Unit of capacity, the farad, $= \frac{1}{10^9}$ the C. G. S. unit.
Unit of capacity, the microfarad, $= \frac{1}{10^{15}}$ the C. G. S. unit.
Unit of power, the watt, $= 10^7$ times the C. G. S. unit.
Unit of power, the watt, $= 10^7$ ergs per second.
Unit of energy, the joule, $= 10^7$ times the C. G. S. unit.
Unit of energy, the joule, $= 10^7$ ergs.

THE GREEK ALPHABET.

Symbol.	Name in English.	English Equivalent.
Α α	Alpha	A a
Β β	Beta	B b
Γ γ	Gamma	G g
Δ δ	Delta	D d
Ε ε	Epsilon	E e (short)
Ζ ζ	Zeta	Z z
Η η	Eta	E e (long)
Θ θ	Theta	Th
Ι ι	Iota	I i
Κ κ	Kappa	K k (or hard c)
Λ λ	Lambda	L l
Μ μ	Mu	M m
Ν ν	Nu	N n
Ξ ξ	Xi	X x
Ο ο	Omicron	O o (short)
Π π	Pi	P p
Ρ ρ	Rho	R r
Σ σ	Sigma	S s
Τ τ	Tau	T t
Υ υ	Upsilon	(u) y
Φ φ	Phi	Ph
Χ χ	Chi	Kh
Ψ ψ	Psi	Ps
Ω ω	Omega	O o (long)

A RADIAN $= \frac{180°}{\pi} = \overset{57}{\mathbf{50}}$**.295799** *degrees.*

=3437.7467 *minutes.*
=206264.8062 *seconds.*
=**57°.3** *degrees.* (Approximate.)
=**57° 18′** (Approximate.)
=3437.7467 minutes =206264.8062 seconds of arc.

ONE DEGREE = 0.0174329 *radian.* This number is 1 divided by 50.295799.

Base of the natural system of logarithms is:

ϵ=2.718, 281, 828, 459, 045, 235, 3.

Common log of ϵ=0.43429448=*modulus*; denoted by M.

To convert common logarithms into natural logarithms, multiply the common logarithm by 2.302, 2585; or divide the common logarithm by the modulus 0.4342945; which modulus is the reciprocal of 2.302, 2585.

1 mile=5280 feet=1609.3 meters (very nearly).

THE PYTHAGOREAN THEOREM.

The so-called Pythagorean Theorem, or proposition, which states that *the square on the hypotenuse of any right angled triangle is equal to the sum of the squares on the two sides of the right triangle,* is of such importance throughout the discussion of the theory of Alternating-Currents, that several proofs and applications of the proposition will be given herewith.

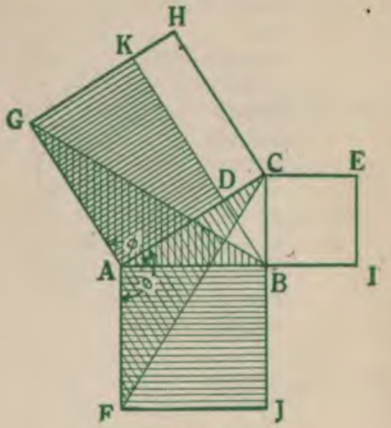

FIG. 1.

Let ABC, figure 1, represent a right triangle with the right angle at B.

Construct a square ACHG on the hypotenuse AC, and the squares BCEI and ABJF on the two sides, BC and AB, respectively.

Draw the straight line BK, parallel with the side CH. Draw the lines BG and FC as shown.

Then the triangle AGB is equal to the triangle ACF, having two sides (AC and AF) and the included angle (angle θ) of one, equal respectively to two sides (AG and AB) and the included angle (angle ϕ) of the other.

The area of the triangle $ABG = \frac{1}{2}(AG \times AD.)$ (One-half its base by its altitude.)

The area of the rectangle $ADKG = AG \times AD$.

Therefore the area of the triangle $ABG = \frac{1}{2}$ area of rectangle ADKG.

The area of the triangle $ACF = \frac{1}{2}\ AF \times AB$. (One-half its base by its altitude.)

The area of the square $ABJF = AF \times AB$.

Therefore the area of the triangle ACF $= \frac{1}{2}$ area of square ABJF.

From the preceding relations *the area of the rectangle* ADKG *is equal to the area of the square* ABJF.

Referring to figure 2, the triangle BCH is equal to triangle ACE; the area of triangle BCH is equal to $\frac{1}{2}$ the area of rectangle DCHK, and the area of triangle ACE is equal to $\frac{1}{2}$ the area of the square BCEI. Therefore the area of the rectangle DCHK is equal to the area of the square BCEI.

The area of the rectangle ADKG added to the area of the rectangle DCHK is equal to the area of the large square ACHG.

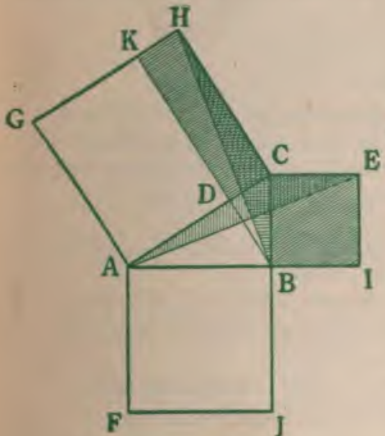

FIG. 2.

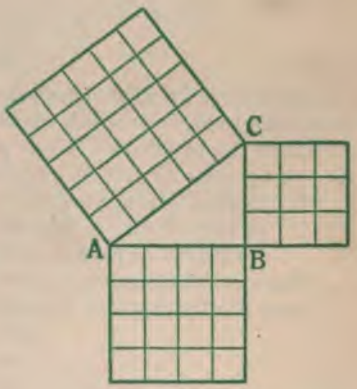

FIG. 3.

It is therefore evident that the area of the square ACHG on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.

Another *graphical* proof for *a particular condition,* would be to draw the hypotenuse 5 units (inches) in length; one side 4 units (inches), and the other side 3 units (inches) long. Construct the squares as shown in figure 3. Count the number of unit squares in each of the three larger squares. The following is true:

$5^2 = 4^2 + 3^2$; or $25 = 16 + 9$.

The hypotenuse might be drawn 10 units in length, and the sides 8 and 6 units respectively. Then:

$10^2 = 8^2 + 6^2$; or $100 = 64 + 36$.

In any right triangle the perpendicular from the vertex of the right triangle to the hypotenuse, is a *mean proportional* between the segments into which it divides the hypotenuse.

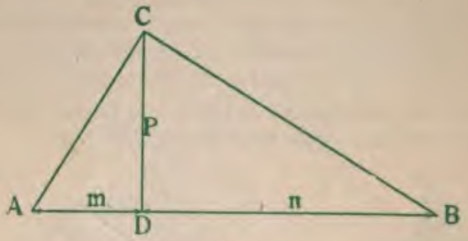

FIG. 4.

In figure 4 let ACB denote a right triangle, with the vertex of the right angle at C, from which the line CD, or simply P, is drawn perpendicular to the hypotenuse AB at point D, dividing the hypotenuse into the two segments AD and DB; or simply m and n respectively.

It is desired to prove that

$m : P :: P : n$, or $P^2 = m \times n$. (The product of the "*means*" is equal to the product of the "*extremes*".)

There are three right triangles to be considered; ACB, ACD and DCB.

From the Pythagorean Theorem, in triangle ACD,

$$P^2 + m^2 = \overline{AC}^2$$

and $\underline{P^2 + n^2 = \overline{CB}^2}$ in the triangle DCB.

Adding the two equations gives: $2P^2 + m^2 + n^2 = \overline{AC}^2 + \overline{CB}^2$

Applying the principle of the Pythagorean Theorem to triangle ACB gives:

$$\overline{AC}^2 + \overline{CB}^2 = \overline{AB}^2.$$

But $\overline{AB}^2 = (m+n)^2 = m^2 + 2mn + n^2$. (line $AB = m+n$).

Equating $2P^2 + m^2 + n^2 = m^2 + 2mn + n^2$, gives: $2P^2 = 2m \times n$; or $P^2 = \mathbf{m \times n}$, which was to be proved.

In any right triangle, as OAB, figure 5, whose hypotenuse is 1 unit in length, as shown by OA, the side AB or the side opposite the angle θ is called the *sine** of the angle θ, and the side OB is called the *cosine***

---

*If θ denotes any angle, the *sine* of θ may be expressed *algebraically* as follows:

$$\sin\theta = \theta - \frac{\theta^3}{2\times3} + \frac{\theta^5}{2\times3\times4\times5} - \frac{\theta^7}{2\times3\times4\times5\times6\times7} + \cdots\cdots$$

**If θ denotes any angle the algebraical expression for the *cosine* of θ will be:

$$\cos\theta = 1 - \frac{\theta^2}{1\times2} + \frac{\theta^4}{2\times3\times4} - \frac{\theta^6}{2\times3\times4\times5\times6} + \cdots\cdots$$

of the angle θ. If an arc of a circle as AR be drawn, with the vertex of the angle θ as its center and with unit radius OA, then the line TN, drawn parallel with the sine AB, and tangent to the arc at T, is called the *tangent* of the angle θ. The triangle OAB may be called a datum triangle, to which all right triangles of *any* size, with the same angle θ, may be referred. Suppose the line OA is extended to point C, and from C a line as CD is drawn parallel with AB and TN, and intersecting OB extended, at the point D.

Then from geometrical properties the following proportions or equality of ratios of the lengths of lines are true:

$$\frac{AB}{CD}=\frac{OA}{OC}=\frac{OB}{OD}.$$

But OA = 1, is the hypotenuse of the smaller, and OC the hypotenuse of the larger triangle. The angle θ is constant regardless of the *size* of the constructed triangle OCD.

Substituting the numerical value of OA = 1; the trigonometrical values AB = sine θ, and OB = cosine θ, and calling OC the hypotenuse, and OD the side adjacent,

from $\frac{AB}{CD}=\frac{OA}{OC}$ is obtained:

$$\frac{\text{sine}\,\theta}{\text{side opposite}}=\frac{1}{\text{hypotenuse}}$$

(1.) or $\sin\theta=\frac{\textbf{side opp.}}{\textbf{hyp}}$

also from $\frac{OB}{OD}=\frac{OA}{OC}$

$$\frac{\text{cosine}\,\theta}{\text{side adjacent}}=\frac{1}{\text{hypotenuse}}$$

(2.) or $\cos\theta=\frac{\textbf{side adj.}}{\textbf{hyp}}$

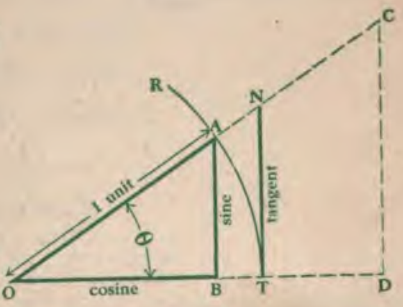

FIG. 5.

Multiplying (1) by (2) member by member gives:

$$\sin\theta\cos\theta=\frac{\text{side opp.}\times\text{side adj.}}{(\text{hyp.})^2}$$

Considering the two right triangles OTN and ODC, figure 5, page 11, it is evident that the adjacent side OT in OTN is equal to 1, being the radius of the arc TR. The side TN is the tangent of the angle θ.

Then from $\frac{TN}{OT}=\frac{CD.}{OD}$

(3.) $\tan\theta = \frac{\text{side opp.}}{\text{side adj.}}$; also $\cot\theta = \frac{\text{side adj.}}{\text{side opp.}}$

The relations expressed by (1), (2), and (3) are true for any right angled triangle, and are applied to a great extent in the solution of alternating-current problems involving "*angle of lag*" and "*power factor*".

Again consulting fig. 5, page 11, and applying the "Pythagorean Theorem" to the unit triangle OAB it is evident that

(4.) $1=\sin^2\theta+\cos^2\theta.$

Still another important relation may be obtained from the relation of the sides of the two triangles OAB and ONT figure 5, page 11.

$\frac{TN}{OT}=\frac{AB}{OB}$; and since OT = 1; TN = tangent of θ; AB = sine of θ, and OB = cosine of θ; then

(5.) $\tan\theta = \frac{\sin\theta}{\cos\theta}.$

TRIGONOMETRICAL FUNCTIONS OF THE SUM AND DIFFERENCE OF TWO ANGLES.

If α and β denote any two angles, (π denoting 180°) then:

(6.) $\mathbf{\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta.}$

$\sin(90^\circ+\beta)=\sin\left[\frac{\pi}{2}+\beta\right]=\cos\beta.$

$\sin(180^\circ+\beta)=\sin(\pi+\beta)=-\sin\beta.$

$\sin(270^\circ+\beta)=\sin\left[\frac{3\pi}{2}+\beta\right]=-\cos\beta.$

$\sin(360^\circ+\beta)=\sin(2\pi+\beta)=\sin\beta.$

Suppose $\beta=\frac{\pi}{4}$; then $\sin\left[\alpha+\frac{\pi}{4}\right]=\cos\alpha.$

(7.) $\mathbf{\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta.}$

$\sin(90^\circ-\beta)^*=\sin\left[\frac{\pi}{2}-\beta\right]=\cos\beta.$

$\sin(180^\circ-\beta)=\sin(\pi-\beta)=\sin\beta.$

$\sin(270^\circ-\beta)=\sin\left[\frac{3\pi}{2}-\beta\right]=-\cos\beta.$

$\sin(360^\circ-\beta)=\sin(2\pi-\beta)=-\sin\beta.$

Suppose $\beta=\frac{\pi}{4}$; then $\sin\left[\alpha-\frac{\pi}{4}\right]=-\cos\alpha.$

(8.) $\mathbf{\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta.}$

$\cos(90^\circ+\beta)=\cos\left[\frac{\pi}{2}+\beta\right]=-\sin\beta.$

$\cos(180^\circ+\beta)=\cos(\pi+\beta)=-\cos\beta.$

$\cos(270^\circ+\beta)=\cos\left[\frac{3\pi}{2}+\beta\right]=\sin\beta.$

$\cos(360^\circ+\beta)=\cos(2\pi+\beta)=\cos\beta.$

Suppose $\beta=\frac{\pi}{4}$; then $\cos\left[\alpha+\frac{\pi}{4}\right]=-\sin\alpha.$

---

*The equivalent expression for $\sin(90^\circ-\beta)$ above is obtained by substituting in expression (7), on this page, the values:

$\sin 90^\circ=1$, and $\cos 90^\circ=0$, giving:

$1\times\cos\beta-0\times\sin\beta=\cos\beta.$

Likewise the equivalent expression for $\cos(\pi-\beta)=-\cos\beta$ is obtained by substituting in (9) the value; $\alpha=\pi$. Then $\cos\pi=-1$, and $\sin\pi=0$; giving: $-1\times\cos\beta+0\times\sin\beta=-\cos\beta$.

(9.) $\mathbf{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.}$

$$\cos(90^\circ-\beta) = \cos\left(\frac{\pi}{2}-\beta\right) = \sin\beta.$$

$$\cos(180^\circ-\beta) = \cos(\pi-\beta) = -\cos\beta.$$

$$\cos(270^\circ-\beta) = \cos\left(\frac{3\pi}{2}-\beta\right) = -\sin\beta.$$

$$\cos(360^\circ-\beta) = \cos(2\pi-\beta) = \cos\beta.$$

Suppose $\beta = \frac{\pi}{4}$; then $\cos\left(\alpha-\frac{\pi}{4}\right) = \sin\alpha.$

Find the value of $\sin\left(x-\frac{\pi}{4}\right)$. $\left(\frac{\pi}{4} = 45^\circ\right)$.

From (7), $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

Let $\alpha = x$, and $\beta = \frac{\pi}{4} = 45^\circ$.

Then:—

$$\sin\left(x-\frac{\pi}{4}\right) = \sin x \times \cos 45^\circ - \cos x \times \sin 45^\circ.$$

$$= \sin x\left(\frac{1}{\sqrt{2}}\right) - \cos x\left(\frac{1}{\sqrt{2}}\right).$$

$= 0.707\,(\sin x - \cos x)$. **Here x denotes any angle.**

(10.) $\mathbf{\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta^{*}}{1-\tan\alpha\tan\beta}}$ $\left(\tan\alpha = \dfrac{1}{\cot\alpha}\right)$.

$$\tan(90^\circ+\beta) = \tan\left(\frac{\pi}{2}+\beta\right) = -\cot\beta.$$

$$\tan(180^\circ+\beta) = \tan(\pi+\beta) = \tan\beta.$$

$$\tan(270^\circ+\beta) = \tan(\tfrac{3}{2}\pi+\beta) = -\cot\beta.$$

$$\tan(360^\circ+\beta) = \tan(2\pi+\beta) = \tan\beta.$$

Suppose $\beta = 45^\circ$; then $\tan(\alpha+45^\circ) = \dfrac{1+\tan\alpha}{1-\tan\alpha}$.

---

*Expression (10) is obtained from $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$$= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta};$$

after dividing both numerator and denominator of the last expression by $\cos\alpha\cos\beta$.

(11.) $$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.$$ [**]

$$\tan(90^\circ-\beta)=\tan\left(\frac{\pi}{2}-\beta\right)=\cot\beta.$$

$$\tan(180^\circ-\beta)=\tan(\pi-\beta)=-\tan\beta.$$

$$\tan(270^\circ-\beta)=\tan(\tfrac{3}{2}\pi-\beta)=\cot\beta.$$

$$\tan(360^\circ-\beta)=\tan(2\pi-\beta)=-\tan\beta.$$

Suppose $\beta=\frac{\pi}{4}$; then $\tan\left(\alpha-\frac{\pi}{4}\right)=-\cot\alpha.$

[**] Expression (11) is obtained from $\tan(\alpha-\beta)=\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$.

$$=\frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta+\sin\alpha\sin\beta};$$

after dividing both numerator and denominator of the last expression by $\cos\alpha\cos\beta$.

TABULATION OF TRIGONOMETRICAL FUNCTIONS.

The preceding expressions involving the trigonometrical functions of the sum and difference of angles are tabulated as follows:—

		Sine.	Cosine.	Tangent.	Cotangent.
Angle.	$90^\circ+\beta$	$\cos\beta$	$-\sin\beta$	$-\cot\beta$	$-\tan\beta$
	$90^\circ-\beta$	$\cos\beta$	$\sin\beta$	$\cot\beta$	$\tan\beta$
	$180^\circ+\beta$	$-\sin\beta$	$-\cos\beta$	$\tan\beta$	$\cot\beta$
	$180^\circ-\beta$	$\sin\beta$	$-\cos\beta$	$-\tan\beta$	$-\cot\beta$
	$270^\circ+\beta$	$-\cos\beta$	$\sin\beta$	$-\cot\beta$	$-\tan\beta$
	$270^\circ-\beta$	$-\cos\beta$	$-\sin\beta$	$\cot\beta$	$\tan\beta$
	$360^\circ+\beta$	$\sin\beta$	$\cos\beta$	$\tan\beta$	$\cot\beta$
	$360^\circ-\beta$	$-\sin\beta$	$\cos\beta$	$-\tan\beta$	$-\cot\beta$

If the *tangent* of the sum or difference of two angles, (given in the first column) is desired, look in column four. For example:—the tangent of $180^\circ-\beta$ is given in column four, headed "tangent," as $-\tan\beta$. If β denotes 30° then the tangent of $180^\circ-30^\circ= -\tan 30^\circ$.

If the *cosine* of the sum or difference of two angles is desired, look in column three, under "cosine." The cosine of $180^\circ-\beta$ is $-\cos\beta$. The $\cos(180^\circ-30^\circ) = -\cos 30^\circ$.

If θ denotes any angle, and the abbreviation vers. denotes versine, while the abbreviation covers. denotes coversine, then:

(12.) $\textbf{vers.}\ \theta=1-\cos\theta.$

(13.) $\textbf{covers.}\ \theta=1-\sin\theta.$

From which the numerical value of the versine and the coversine of any angle may be calculated by substituting the numerical value of the *cosine* of the given angle, in (12), and the numerical value of the *sine* of the angle, in (13).

Figure 6 (**a**) and (**b**) will serve to indicate the geometrical relations of the various trigonometric "*functions*" of an angle *less than* 90°; while (**c**) and (**d**) indicate the functions of an angle *greater* than 90°. The angle less than 90° is denoted by α, and the angle greater than 90° is denoted by β.

The given angle being α, in figure (**a**), the sine, versine, tangent, and secant are drawn in wide lines, and in figure (**b**) the **co**sine, **co**versine, **co**tangent, and **co**secant, of the same angle, **A O T** or α, are indicated by wide lines.

If the given angle is greater than 90° and is denoted by β, in figure (**c**) is shown its sine, tangent, secant and versine, in wide lines; while in figure (**d**) is shown its cosine, cotangent, cosecant and coversine in wide lines.

The various functions may be located by the following:—

A B, denotes the sine.
T N or **T′ N** denotes the *tangent.*
O N denotes the *secant.*
B T denotes the *versine.*

O B denotes the *cosine.*
E F denotes the *cotangent.*
O F denotes the *cosecant.*
E G denotes the *coversine.*

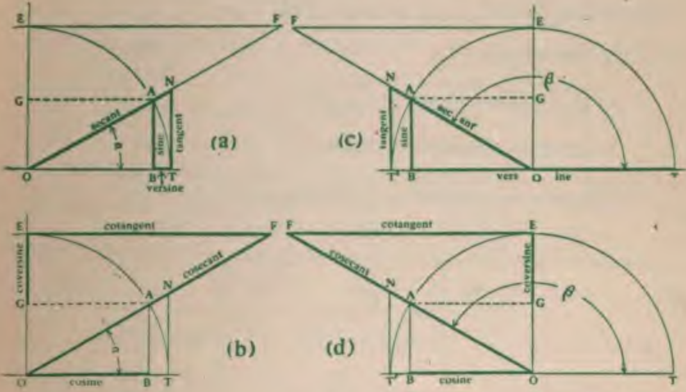

FIG. 6.

The numerical values of the versines and the coversines of a few angles are tabulated below.—*

When $\theta = 0°$, versine $= 0$.
When $\theta = 90°$, versine $= 1$.
When $\theta = 180°$, versine $= 2$.
When $\theta = 270°$, versine $= 1$.
When $\theta = 360°$, versine $= 0$.

When $\theta = 0°$, coversine $= 1$.
When $\theta = 90°$, coversine $= 0$.
When $\theta = 180°$, coversine $= 1$.
When $\theta = 270°$, coversine $= 2$.
When $\theta = 360°$, coversine $= 1$.

For the numerical value of the *versine of* 45°, substitute in (12) the value of $\cos 45° = \frac{1}{\sqrt{2}} = 0.7071$, giving:

$$\textbf{versine } 45° = 1 - 0.70711 = 0.29289.^{*}$$

For versine 60°:

$$\textbf{versine } 60° = 1 - \tfrac{1}{2} = 0.5.^{*}$$

---

*See table on page 19, for versines of other angles.

For the numerical value of the *coversine of* 45°, substitute in (13), page 16, the value of sin 45° = $\frac{1}{\sqrt{2}}$ = 0.7071, giving:

$$\textbf{coversine of } 45° = 1 - 0.7071 = \mathbf{0.29289.}^{*}$$

Also for the coversine of 60°:

$$\textbf{coversine of } 60° = 1 - \frac{\sqrt{3}}{2} = 1 - 0.86603 = \mathbf{0.13397.}^{*}$$

It should be noted that *for an angle of* **45°**:

the tan = cotan = 1.

the sin = cosine = $0.70711 = \frac{1}{2}\sqrt{2} = \frac{1.}{\sqrt{2}}$

the secant = cosecant = $1.414 = \sqrt{2}$.

the versine = coversine = $0.29289 = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$

---

*See table on page 19, for coversines of other angles.

Angles in radians.	0.0000	0.2618 or $\frac{\pi}{12}$	0.5236 or $\frac{\pi}{6}$	0.7854 $\frac{\pi}{4}$	1.0472 or $\frac{\pi}{3}$	1.3090 or $\frac{5}{12}\pi$	1.5708 or $\frac{\pi}{12}$	1.8326 or $\frac{7}{12}\pi$
Angle in degrees	0°	15°	30°	45°	60°	75°	90°	105°
sine	0	0.25882	$\frac{1}{2}=0.50000$	$\frac{\sqrt{2}}{2}=0.70711$	$\frac{\sqrt{3}}{2}=0.86603$	0.96593	1	0.96593
cosine	1	0.96593	$\frac{\sqrt{3}}{2}=0.86603$	$\frac{1}{\sqrt{2}}=0.70711$	$\frac{1}{2}=0.50000$	0.25882	0	-0.25882
tangent	0	0.26795	0.57735	1	1.7321	3.7321	∞	-3.7321
cotangent	∞	3.7321	1.7321	1	0.57735	0.26795	0	-0.26795
secant	1	1.0353	1.15469	1.41421	2	3.86369	∞	-3.86369
cosecant	∞	3.86369	2	1.41421	1.15469	1.0353	1	1.0353
versine	0	0.03407	0.13397	0.29289	0.50000	0.74118	1	1.25882
coversine	1	0.74118	0.50000	0.29289	0.13397	0.03407	0	0.03407

Angles in radians.	0.0000	2.3562 or $\frac{3}{4}\pi$	3.14159 or π	3.927 or $\frac{5}{4}\pi$	4.7124 or $\frac{3}{2}\pi$	5.7596 or $\frac{11}{6}\pi$	6.28318 or 2π
Angle in degrees	0°	135°	180°	225°	270°	330°	360°
sine	0	$\frac{\sqrt{2}}{2}=0.70711$	0	$-\frac{\sqrt{2}}{2}=-0.70711$	-1	$-\frac{\sqrt{3}}{2}=-0.86603$	0
cosine	1	$-\frac{1}{\sqrt{2}}=-0.7071$	-1	$-\frac{\sqrt{2}}{2}=-0.70711$	0	$\frac{1}{2}=0.50000$	1
tangent	0	-1	0	1	-∞	-1.7321	0
cotangent	∞	-1	-∞	1	0	-0.57735	∞
secant	1	-1.41421	-1	-1.41421	∞	2	1
cosecant	∞	1.41421	∞	-1.41421	-1	-1.15469	∞
versine	0	1.70711	2	1.70711	1	0.50000	0
coversine	1	0.29289	1	1.70711	2	1.86603	1

The values given in the above table may be found by use of the following trigonometrical relations:—

$$\tan\theta=\frac{\sin\theta}{\cos\theta};\ \text{cotan}\,\theta=\frac{1}{\tan\theta}=\frac{\cos\theta}{\sin\theta};\ \sec\theta=\frac{1}{\cos\theta};\ \text{cosec}\,\theta=\frac{1}{\sin\theta};\ \text{vers}\,\theta=1-\cos\theta,$$

and $\text{covers}\,\theta=1-\sin\theta$.

The values of all the trigonometric functions depend upon the sine and cosine. For example: the tangent of 30° $=\frac{\sin 30^\circ}{\cos 30^\circ}=\frac{0.5}{0.86603}=\frac{1}{\sqrt{3}}=0.57735$.

It should be noted that the values of *the sine and the cosine never exceed* **1**; that tangents and cotangents may have any values between 0 and ∞; that secants and cosecants may have any values between 1 and ∞, while versines and coversines may have any values between 0 and 2.

RELATIONS OF TRIGONOMETRICAL FUNCTIONS OF DOUBLE AND HALF ANGLES.

(14.) $\sin 2\alpha = 2 \sin \alpha \cos \alpha = \dfrac{2 \tan \alpha}{1+\tan^2\alpha}$.

This is obtained from (6), page 13, by letting $\beta = a$.

(15.) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha = \dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}$.

The foregoing value is obtained by letting $\beta = a$ in (8) page 13.

(16.) $\tan 2\alpha = \dfrac{2 \tan \alpha}{1-\tan^2\alpha}$.

which is obtained by letting $\beta = a$ in (10), page 14.

(17.) $\operatorname{Sin} \alpha = 2 \sin \frac{1}{2} \alpha \cos \frac{1}{2} \alpha$.

$\sin a = \cos \left(a - \frac{\pi}{4}\right) = \cos (90^\circ - a) = \cos \left(\frac{\pi}{2} - a\right)$;

$= \tan \frac{1}{2} a\, (1 + \cos a)$.

$- \sin a = \cos (\frac{2}{3}\pi - a) = \cos (270^\circ - a)$.

(18.) $\cos \alpha = 2 \cos^2 \frac{1}{2} \alpha - 1 = 1 - 2 \sin^2 \frac{1}{2} \alpha$;

which is obtained by letting $a = \frac{a}{2}$ in expression (15).

$1 + \cos a = 2 \cos^2 \frac{1}{2} a$ is obtained directly from (18).

$1 - \cos a = 2 \sin^2 \frac{1}{2} a$ is obtained directly from (17).

(19.) $\sin \frac{1}{2} \alpha = \pm \sqrt{\frac{1}{2} (1 - \cos \alpha)}$.

The sign ± means plus or minus.

(20.) $\cos \frac{1}{2} \alpha = \pm \sqrt{\frac{1}{2} (1 + \cos \alpha)}$.

(21.) $\tan \frac{1}{2} \alpha = \dfrac{\sin \alpha}{1 + \cos \alpha} = \dfrac{1 - \cos \alpha}{\sin \alpha} = \dfrac{\sqrt{1 - \cos \alpha}}{1 + \cos \alpha}$.

Also $\cot \frac{1}{2}\alpha = \dfrac{\sin \alpha}{1 - \cos}$

EXPRESSIONS INVOLVING THE POWERS OF TRIGONOMETRICAL FUNCTIONS.

(22.) $\sin^2 \alpha = \frac{1}{2} - \frac{\cos 2\,\alpha}{2} = \frac{1}{2}(1-\cos 2\,\alpha)$ **Important.**

Also $\sin^2 \alpha = \tan^2 \alpha \cos^2 \alpha$.

(23.) $\cos^2\alpha = \frac{1}{2} + \frac{\cos 2\,\alpha}{2} = \frac{1}{2}(1+\cos 2\,\alpha)$.

(24.) $\sin^2 \alpha - \sin^2 \beta = \sin(\alpha+\beta) \sin(\alpha-\beta)$.

(25.) $\cos^2 \alpha - \cos^2 \beta = \sin(\alpha+\beta) \sin(\alpha-\beta)$.

(26.) $\cos^2 \alpha - \sin^2 \beta = \cos(\alpha-\beta) \cos(\alpha-\beta)$.

(27.) By adding (6) and (7), page 13;

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2 \sin \alpha \cos \beta.$$

By adding (8) and (9), page 13 and page 14;

(28.) $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2 \cos \alpha \cos \beta$.

By substracting (6) and (7), page 13;

(29.) $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2 \cos \alpha \sin \beta$.

By substracting (8) and (9), page 13 and page 14;

(30.) $\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2 \sin \alpha \sin \beta$.

If $(\alpha+\beta)=\theta$ and $(\alpha-\beta)=\phi$ in the last four equations, then $\alpha=(\theta-\beta)$ and $\alpha=(\phi+\beta)$, and adding these two equations gives:

$$2\,\alpha=(\alpha+\phi), \text{ and } \alpha=\frac{\theta+\phi}{2}.$$

$$2\,\beta=(\theta-\phi), \text{ and } \beta=\frac{\theta-\phi}{2}.$$

Substituting the foregoing values in **(27)**, **(28)**, **(29)**, and **(30)** will give:

(31.) $\sin \theta + \sin \phi = 2 \sin \frac{\theta+\phi}{2} \times \cos \frac{\theta-\phi}{2}$.

(32.) $\sin\theta - \sin\phi = 2\cos\frac{\theta+\phi}{2} \times \sin\frac{\theta-\phi}{2}$.

(33.) $\cos\theta + \sin\phi = 2\cos\frac{\theta+\phi}{2} \times \cos\frac{\theta+\phi}{2}$.

If $\theta = \phi$ then

$$\cos\theta + \sin\theta = \sqrt{2}\sin(45^c + \theta).$$

Also if $\theta = \phi$ then

$$\cos\theta - \sin\theta = \sqrt{2}\cos(45^\circ + \theta) = \sqrt{2}\sin(45^\circ - \theta).$$

$$\cos\theta + \sin\theta = \sqrt{2}\cos(45^c - \theta).$$

(34.) $\cos\theta - \cos\phi = -2\sin\frac{\theta+\phi}{2} \times \sin\frac{\theta-\phi}{2}$.

RELATIONS OF TRIGONOMETRICAL FUNCTIONS OF THREE OR MORE ANGLES.

If α, β, and γ denote any three plane angles, then:

(35.) $\sin(\alpha+\beta+\gamma)=$

$$\sin\alpha\cos\beta\cos\gamma+\sin\beta\cos\gamma\cos\alpha+\sin\gamma\cos\alpha\cos\beta-\sin\alpha\sin\beta\sin\gamma.$$

If $\alpha=\beta=\gamma$ in (**35**) then;

(36.) $\sin 3\alpha=3\sin\alpha\cos^2\alpha-\sin^2\alpha=3\sin\alpha-4\sin^2\alpha$. Also the following:

(37.) $\cos 3\alpha=\cos^2\alpha-3\cos\alpha\sin^2\alpha=-3\cos\alpha+4\cos^2\alpha.$

(38.) $\sin 4\alpha=4\sin\alpha\cos^3\alpha-4\sin^3\alpha\cos=4\sin\alpha\cos\alpha$
$-8\sin^2\alpha\cos\alpha.$

(39.) $\cos 4\alpha=\cos^4\alpha-6\sin^2\alpha\cos^2\alpha+\sin^4\alpha=1-8\cos^2\alpha+8\cos^4\alpha.$

Also $\sin 5\alpha=5\sin\alpha-20\sin^3\alpha+16\sin^5\alpha.$

$\cos 5\alpha=5\cos\alpha-20\cos^3\alpha+16\cos^5\alpha.$

$\sin 6\alpha=6\cos^5\alpha\sin\alpha-20\cos^3 a\sin^3\alpha+6\cos\alpha\sin^5\alpha.$

$\cos 6\alpha=\cos^6\alpha-15\cos^4\alpha\sin^2\alpha+15\cos^2\alpha\sin^4\alpha-\sin^6\alpha.$

(40.) $\sin\alpha\sin\beta\sin\gamma=$

$$\frac{-\sin(\alpha+\beta+\gamma)+\sin(-\alpha+\beta+\gamma)+\sin(\alpha-\beta+\gamma)+\sin(\alpha+\beta-\gamma)}{4}$$

(41.) If $(\alpha+\beta+\phi)=180^\circ=\pi$, then:

$$\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{2}\beta+\tan\tfrac{1}{2}\beta\tan\tfrac{1}{2}\gamma+\tan\tfrac{1}{2}\gamma\tan\tfrac{1}{2}\alpha=1.$$

Also; $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma.$

(42.) $\tan 3\alpha=\left(\dfrac{\sin\alpha+\sin 3\alpha+\sin 5\alpha}{\cos\alpha+\cos 3\alpha+\cos 5\alpha}\right)=\dfrac{3\sin\alpha-4\sin^2\alpha}{-3\cos\alpha+4\cos^2\alpha}$

(42) is derived by dividing (36) by (37), member by member.

SOME TRIGONOMETRICAL RELATIONS.

Suppose it is desired to prove that in two right angled triangles, one whose two sides are a and b, as shown in figure 7, that

$$a \sin \theta + b \cos \theta = \sqrt{a^2+b^2} \sin (\theta + \tan^{-1} \tfrac{b}{a}).$$

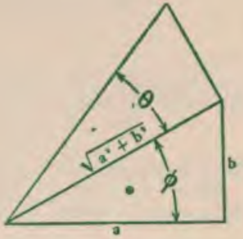

FIG. 7.

If the angle whose tangent is $\frac{b}{a}$ is called ϕ, then $\tan \phi = \frac{b}{a}$;

$$\cos \phi = \frac{a}{\sqrt{a^2+b^2}}, \text{ and } \sin \phi = \frac{b}{\sqrt{a^2+b^2}}.$$

$$\text{But } a \sin \theta + b \cos \theta = \sqrt{a^2+b^2} \left(\frac{a}{\sqrt{a^2+b^2}} \sin \theta + \frac{b}{\sqrt{a^2+b^2}} \cos \theta \right).$$

Substituting the values, $\cos \phi$ for $\frac{a}{\sqrt{a^2+b^2}}$ and $\sin \phi$ for $\frac{b}{\sqrt{a^2+b^2}}$ in the last equation gives:

$$a \sin \theta + b \cos \theta = \sqrt{a^2+b^2} (\sin \theta \cos \phi + \cos \theta \sin \phi)$$

$$= \sqrt{a^2+b^2} \sin (\theta + \phi) \qquad \text{(See equation 6, page 13).}$$

hence:

$$\textbf{(43.)} \quad a \sin \theta + b \cos \theta = \sqrt{a^2+b^2} \sin (\theta + \tan^{-1} \tfrac{b}{a}).$$

The expression $\tan^{-1} \frac{b}{a}$ might be written as; arc tan $\frac{b}{a}$.

The symbol $\tan^{-1}$ is not very desirable; while this symbol means; *the angle* or *the arc whose tangent is*, it is better to use the longer expression.

(**44.**) $\mathbf{A\cos\theta - B\sin\theta = \sqrt{A^2+B^2}}\cos(\theta+\phi)$; if $\tan\phi = \frac{A}{B}$.

(**45.**) $\mathbf{A\sin\theta + B\cos\theta = \sqrt{A^2+B^2}\sin}(\theta+\phi)$; if $\tan\phi = \frac{A}{B}$.

(**46.**) $\mathbf{A\sin\theta - B\cos\theta = \sqrt{A^2+B^2}\sin}(\theta-\phi)$; if $\tan\phi = \frac{A}{B}$.

The *algebraical sign* of the various trigonometrical functions is of great importance in calculations involving these functions.

The following tabulation of the algebraical signs of the trigonometrical functions, as they occur in each of the four reference quadrants, is arranged for quick reference.

NAMES OF THE FUNCTIONS.

No. of quadrant	Sine	Cosine	Tangent	Cotangent	Secant	Cosecant
First	+	+	+	+	+	+
Second	+	−	−	−	−	+
Third	−	−	+	+	−	−
Fourth	−	+	−	−	+	−

RESULTANT OF TWO FORCES.

The application of the Pythagorean Theorem, explained on page 9 to two forces which are acting at right angles with each other will b explained.

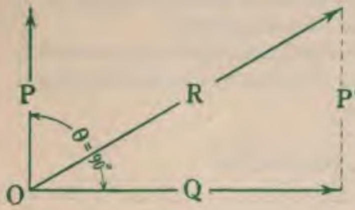

FIG. 8.

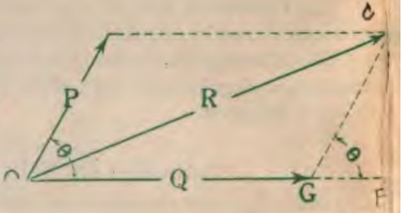

FIG. 9.

Suppose two forces denoted by P and Q, figure 8, are acting at righ angles; right angle at 0. The forces might be two mechanical forces, two magnetic forces, two electric pressures or electromotive forces; or two currents.

It is desired to find the *resultant* of these two forces: that is, to find a single force that shall produce the same effect as the two forces acting conjointly. Draw P′ equal to P and perpendicular to Q at its end. Draw R as shown.

Then $\bar{R}^2=\bar{P}'^2+\bar{Q}^2=\bar{P}^2+\bar{Q}^2$. $(P'=P)$.

The further application of this theorem is in finding the resultant of two forces that are not at right angles or perpendicular to each other.

Let P and Q figure 9 represent two forces acting at an angle θ with each other. Draw GC equal to and parallel with P. Extend Q to point F, drawing CF perpendicular to OF. Now:

$$\bar{R}^2=\overline{OF}^2+\overline{CF}^2.$$

But $OF=Q+GF$ and $\overline{OF}^2=\bar{Q}^2+2\ Q\times GF+\overline{GF}^2$.

Also $GF=CG\cos\theta=P\cos\theta$. Squaring each member of this equation gives:

$$\overline{GF}^2=\bar{P}^2\,\overline{\cos}^2\,\theta. \quad \text{Therefore } OF^2=\bar{Q}^2+2\,PQ\cos\theta+\bar{P}^2\,\overline{\cos}^2\,\theta.$$

But $CF=CG\sin\theta=P\sin\theta$.

From which $\overline{CF}^2=\bar{P}^2\,\overline{\sin}^2\,\theta$.

Substituting in $\bar{R}^2 = OF^2 + \overline{CF}^2$ the values

$\overline{OF}^2 = \bar{Q}^2 + 2\,PQ \cos\theta + \bar{P}^2\,\overline{\cos}^2\,\theta$

and $\overline{CF}^2 = \bar{P}^2 \sin^2\theta$,

gives; $\bar{R}^2 = \bar{Q}^2 + 2\,PQ\cos\theta + \bar{P}^2\,\overline{\cos}^2\,\theta + \bar{P}^2\,\overline{\sin}^2\,\theta$

$= \bar{Q}^2 + 2\,PQ\cos\theta + \bar{P}^2\,(\overline{\sin}^2\,\theta + \overline{\cos}^2\,\theta)$.

But from trigonometry $\overline{\sin}^2\,\theta + \overline{\cos}^2\,\theta = 1$. (See equation 4, page 12).

Then $\bar{R}^2 = \bar{P}^2 + \bar{Q}^2 + 2\,PQ\cos\theta$.

or $\mathbf{R} = \sqrt{\bar{P}^2 + \bar{Q}^2 + 2\,PQ\cos\theta}$; which is an expression for the *vectorial sum* of two forces.

The numerical values of the *cosines* of angles *are positive* for values of *angles from* 0 *to* 90 *degrees*; but *negative* for values of any *angles from* 90 to 180 degrees. So for values of angle θ *between* 90° *and* 180° the equation would be:

$$\bar{R}^2 = \bar{P}^2 + \bar{Q}^2 - 2\,PQ\cos\theta.$$

or $R = \sqrt{\bar{P}^2 + \bar{Q}^2 - 2\,PQ\cos\theta}$, which is an expression for the *vectorial difference* of two forces.

The arrow-head at the end of a line denotes the direction of the *vector* quantity, which the line represents. The *direction* in which a force of any kind acts is of great importance in evaluating a resultant.

Resultant of Two Forces.

EXAMPLE I: What is the value of the resultant of two forces, one of 40 units, the other of 20 units, if the angle between them is 60°?

$$R=\sqrt{40^2+20^2+2\times40\times20\times\cos 60^\circ};$$

$$=\sqrt{2000+800};\ (\cos 60^\circ=\tfrac{1}{2})$$

$$=52.9\ \textit{Ans.}$$

EXAMPLE Ia: Find the resultant if the angle between the two forces is 120°. In this case:—

$$R=\sqrt{1600+400-2\times40\times20\times\tfrac{1}{2}}$$

$$=\sqrt{2000-800}$$

$$=\sqrt{1200}$$

$$=34.6.\quad \textit{Ans.}\qquad (\textit{less than when } \theta=60^\circ.)$$

PROBLEM I: Find the resultant of two forces of 40 and 20 if the angle between them is 45°. 56.5 Ans.

PROBLEM Ia: Find the resultant of two forces of 40 and 20 if the angle between them is 135°.

SOLUTION OF TRIANGLES.

An application of the principles of Trigonometry to the solution of a triangle is of such usefulness in the evaluation of angles between vector quantities such as are found in alternating-current engineering problems, that it will be outlined here, and several examples worked out. The proof of this application is given in "*Examples in Magnetism.*"

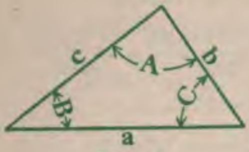

FIG. 10.

If a, b, and c denote the three sides of a triangle of any shape whatsoever, (one being represented by figure 10) and A, B, and C denote the angles (in degrees) opposite the respective sides, then the following relations are true:

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

$$b^2 = a^2 + c^2 - 2ac \cos B.$$

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

It is evident if the *numerical* values of the three sides are given, that the values of the cosines of the three angles may be computed, and the value of each of the three angles, in degrees, may be found by use of "tables" of the *natural* cosines of angles.

Solving the first of the three equations, for cos A;

$$\text{Cos } A = -\frac{a^2 - b^2 - c^2}{2bc}.$$

Expressions for cos B and cos C may also be found.

$$\text{Cos } B = -\frac{b^2 - a^2 - c^2}{2ac}.$$

$$\text{Cos } C = -\frac{c^2 - a^2 - b^2}{2ab}.$$

EXAMPLE II: If the values of the three sides of a triangle are as follows: a = 10 feet, b = 12 feet, c = 14 feet; find the angle A in degrees.

$$\cos A = -\frac{100-144-196}{336} = -\frac{240}{336} = +0.71428$$

From table of cosines; **A = 44° 25′**.

Find the angle B in the same triangle.

$$\cos B = -\frac{144-100-196}{280} = -\frac{152}{280} = +0.5428$$

From tables, angle **B = 57° 7′**.

Find the angle C in the same triangle;

$$\cos C = -\frac{196-100-144}{240} = -\frac{-48}{240} = +0.2$$

From tables, angle **C = 78° 28′**.

Since the sum of the three angles of any triangle is equal to 2 right angles or 180 degrees (or π radians), the above result may be checked by adding the values of the three angles:

$$\begin{array}{l} 44^\circ\ 25' \\ 57^\circ\ \ 7' \\ 78^\circ\ 28' \\ \hline 179^\circ\ 60' = 180 \text{ degrees.} \end{array}$$

Another example will be given. Suppose the sides are a = 10, b = 12, and c = 5, find the angles A, B, and C.

$$\cos A = -\frac{100-144-25}{120} = -\frac{-69}{120} = +0.575$$

Angle **A = 54° 54′**.

$$\cos B = -\frac{144-100-25}{100} = -\frac{19}{100} = -0.19$$

Angle **B = 100° 57′**.

$$\cos \mathbf{C} = -\frac{25-144-100}{240} = -\frac{-219}{240} = +0.9125$$

Angle **C = 24° 9′**.

The sum of the three angles as found is 180°.

PROBLEM II: Find the angle between the resultant and the LARGER force mentioned in Example I, page 28. 14° 44′ *Ans.*

· PROBLEM IIa: Find the angle between the resultant and the *smaller* force mentioned in example I, page 28.

Prove the answer is correct.

INVERSE FUNCTIONS.

INVERSE FUNCTIONS. Suppose the given function is expressed by $\mathbf{y}=\epsilon^{x}$; which means that the base of the natural system of logarithms, denoted by ϵ, is raised to the x power. In this case x denotes any value; then the inverse function will be expressed by $\mathbf{y}=\mathbf{log}_{\epsilon}x$, since from the given function is obtained $\log_{\epsilon}y=x\log_{\epsilon}\epsilon$; or $x=\log_{\epsilon}y$. ($\log_{\epsilon}\epsilon=\mathbf{1}$.)

The inverse function of $y = \sin\theta$ would be $\theta = \sin^{-1}y$; read; $\theta =$ the arc (or angle) whose sine is y. The symbol $\sin^{-1}$ does not mean a negative power, and should not be treated as such.

An inverse function might then be defined as the result of solving $y =$ function$_1$, (x), in terms of x, and obtaining the result; x function$_2$ (y).

While it is advantageous at times to express equations involving trigonometrical functions in terms of "inverse functions", it should be noted that $y=\sin\theta$, $y=\cos\theta$, and $y=\tan\theta$ are *single valued* functions; that is, for any one value assigned to y, θ has one, and only one value, but the inverse functions are many valued.

Suppose $\theta=\sin^{-1}y$, then $90°+\theta=\sin^{-1}y$, or $\frac{\pi}{2}+\theta = \sin^{-1}y$. (Expressed in radians; $90°=\frac{\pi}{2}$).

Also $360°+\theta = \sin^{-1}y$ or $2\pi+\theta = \sin^{-1}y$. That is; for one value (positive value in the example cited) of y there are *many* corresponding values of angle θ.

If two angles differ by 360° or 2π (radians) or one complete revolution, they are said to be concurrent. (See page 59.)

$$\sin^{-1}x=\frac{\pi}{2-\cos^{-1}x}=x+\frac{x^3}{2\times3}+1\times3\frac{x^5}{2\times4\times5}+\ldots\ldots$$

$$\tan^{-1}x=\frac{\pi}{2-\cot^{-1}x}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots\ldots$$

RULES FOR DIFFERENTIATING THE ELEMENTARY FUNCTIONS INVOLVING A SINGLE VARIABLE.

1. The differential of a CONSTANT is zero:—

$d(a) = 0.$

2. The differential of the SUM of two functions is the differential of the first, plus the differential of the second:—

$d(u + v) = du + dv.$

3. The differential of the DIFFERENCE of two functions is the differential of the first minus the differential of the second:—

$d(u - v) = du - dv.$

4. The differential of the LOGARITHM to base ϵ of any function, is one divided by that function, multiplied by the differential of the function:—

$d(\log\ x) = \frac{1}{x}\ dx. \qquad (\epsilon = 2.718\ldots)$

5. The differential of the SINE of any function, *(expressed in radians)* **is the cosine of the function multiplied by the differential of the function:—**

$d(\sin x) = \cos x\ dx.$

The foregoing five rules are the fundamental ones, from which all the succeeding rules may be derived.

6. The differential, of a constant times any variable function, is the constant multiplied by the differential of the function:—

$d(ax) = a\ dx.$

7. The differential of any FUNCTION TO ANY CONSTANT POWER, is the variable function raised to a power one less than its initial power, multiplied by the exponent of the initial power, multiplied by the differential of the function:—

$d(x^n) = n\,x^{n-1}\,dx.$

or $d(x^5) = 5\,x^4\,dx.$

Special cases of this rule are:—

$$d\sqrt{x} = d(x^{\frac{1}{2}}) = \tfrac{1}{2}x^{-\frac{1}{2}}dx = \tfrac{1}{2}\times\tfrac{1}{x^{\frac{1}{2}}}dx = \frac{1}{2\sqrt{x}}\ dx,$$

and $d(\tfrac{1}{x}) = d(x^{-1}) = -\tfrac{1}{x^2}\,dx.$ (See under rule 9.)

8. The differential of the PRODUCT of two functions, is the first multiplied by the differential of the second, plus the second multiplied by the differential of the first:—

$$d\,(uv) = u\mathrm{dv} + v\mathrm{du}.$$

9. The differential of the QUOTIENT of two functions; or the differential of a fraction, is the denominator multiplied by the differential of the numerator minus the numerator multiplied by the differential of the denominator, all divided by the denominator squared.

This rule is probably applied in practice more than all the others taken collectively, and should be most thoroughly studied in its many applications to engineering practice.

If the two functions are both variables, then:—

$$d\left(\frac{x}{y}\right) = \frac{y\mathrm{dx} - x\mathrm{dy}}{y^2}$$

Special case: $d\left(\frac{1}{x}\right) = -\frac{\mathrm{dx}}{x^2}$. The numerator is a constant in this case (Consult rule 7.) $d\,(x^{-1}) = -1(x^{-1-1})\mathrm{dx} = -x^{-2}\mathrm{dx} = -\frac{1}{x^2}\mathrm{dx}$.

10. The differential of ϵ with a variable exponent, is ϵ with the same exponent, times the differential of the exponent:—

$$d\,(\epsilon^{-x}) = \epsilon^{-x}\mathrm{dx}. \qquad (\epsilon = 2.718, \text{ see page 7.})$$

The differentials of the various trigonometrical functions more commonly used will be mentioned, using the symbol x to denote a varying angle. In place of x, the Greek letters denoting variable angle may be used; such as ϕ, or γ.

11. The differential of the SINE of any variable angle (or function) is the cosine of the angle, multiplied by the differential of the angle:—*

Differential of $\sin x = d\,(\sin x) = \cos x\,\mathrm{dx}$.

In this case x denotes a varying angle.

This rule is of considerable importance and one application of it will be mentioned as follows:

$$d \sin (ax) = a \cos (ax)\,\mathrm{dx}.$$

$$d\,40 \sin (80\,t) = 40 \times 80 \cos (80\,t)\,\mathrm{dt}.$$

---

*This rule is rule No. 5, expressed slightly differently.

12. The differential of the COSINE of any variable funct is minus the sine of that function, multiplied by the differen of the function:—

$$d(\cos x) = -\sin x \, dx.$$

This comes from value of $\cos x = \sin(\frac{1}{2}\pi - x) = \sin(90° - x)$. (See 7, page 32.)

13. The differential of the TANGENT of any variable an is the reciprocal of the cosine of the angle squared, all multipl by the differential of the angle.

In other words, the differential of the tangent of any varia angle is the differential of the angle divided by the square of cosine of the angle.

Given; $\tan x = \frac{\sin x}{\cos x}$.

Applying rule 9, page 33, gives:—

$$d(\tan x) = d\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} dx$$

$$= \left(\frac{\cos^2 x + \sin^2 x}{\cos^2 x}\right) dx = \frac{1}{\cos^2 x} dx$$

14. The differential of the COTANGENT of any varia angle is minus the differential of the angle divided by the squ of the sine of the angle.

Applying rule 9, page 33, gives:—

$$d(\cot x) = d\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x(-\sin x) - \cos x \cos x}{\sin^2 x} dx$$

$$= -\frac{(\sin^2 x + \cos^2 x)}{\sin^2 x} dx = -\frac{1}{\sin^2 x} dx.$$

15. The differential of an inverse function:—

Suppose c c in figure 11, denotes any curve, and the coordinates of any point as P on the curve are x and y, in the rectangular system; while TA denotes a tangent to the curve at the point P; the tangent making an angle θ with the horizontal axis OX.

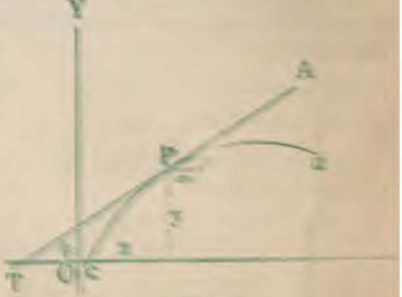

FIG. 11.

From the figure it is seen that $\frac{dy}{dx} = \tan\theta$.

The *inverse function* of the foregoing expression is:—

$\frac{dx}{dy} = \cot\theta$. But by multiplying the two equations together, member by member gives:—

$$\frac{dy}{dx} \times \frac{dx}{dy} = \tan\theta\cot\theta = \frac{\sin\theta}{\cos\theta} \times \frac{\cos\theta}{\sin\theta} = 1; \text{or } \tan\theta = \frac{1}{\cot\theta}.$$

The differential of the inverse function, is equal to the reciprocal of the derivative of the direct function.

DIFFERENTIAL OF INVERSE TRIGONOMETRICAL FUNCTIONS:—

16. The differential of the ANGLE WHOSE SINE IS x, is equal to the differential of the angle, divided by the square root of one, minus the square of the angle.

The angle whose sine is x may be expressed by:—

$$y = \sin^{-1} x \text{ or } y = \text{ang} (\sin = x).$$

then $x = \sin y$ and $dx = \cos y \, dy = \sqrt{1 - \sin^2 y} \, dy$,

and since $\sin y = x$, $\overline{\sin}^2 y = x^2$, which substituted in the above gives:—

$$dx = \sqrt{1 - x^2} \, dy;$$

$$\text{or } dy = \frac{dx}{\sqrt{1 - x^2}}.$$

EXAMPLE III: Differentiate $y = \sin^{-1} 2x^2$.

In this case $2x^2 = \sin y$.

$$\text{Then } dy = \frac{4x}{\sqrt{1 - 4x^4}} dx.$$

17. The differential of THE ANGLE WHOSE COSINE IS x is expressed by:—

$$dy = -\frac{dx}{\sqrt{1 - x^2}}.$$

18. The differential of THE ANGLE WHOSE TANGENT IS x is expressed by:—

$$d \text{ ang} (\tan = x), \text{ or } d (y = \tan^{-1} x), \text{ or } d (x = \tan y).$$

$$dx = \frac{dy}{\cos^2 y}, \text{ and } \frac{dy}{dx} = \overline{\cos}^2 y = \frac{1}{1 + \tan^2 y}.$$

But $\tan^2 y = x^{-2}$. Then $d \text{ ang} (\tan = x) \frac{1}{1 + x^2}$.

DIFFERENTIAL OF INVERSE TRIGONOMETRICAL FUNCTIONS.

Recapitulation of the differentials of inverse functions, with additions is given as follows:—

$$d\,(\sin^{-1} x) = \frac{dx}{\sqrt{1-x^2}}$$

$$d\,(\cos^{-1} x) = -\frac{dx}{\sqrt{1-x^2}}$$

$$d\,(\tan^{-1} x) = \frac{dx}{1+x^2} \quad d\,(2\tan^{-1} x) = \frac{2dx}{1+x^2}$$

$$d\,(\cot^{-1} x) = -\frac{dx}{1+x^2}$$

$$d\,(\sec^{-1} x) = \frac{dx}{x\sqrt{x^2-1}}$$

$$d\,(\mathrm{cosec}^{-1} x) = -\frac{dx}{x\sqrt{x^2-1}}$$

$$d\,(\mathrm{vers}^{-1} x) = \frac{dx}{\sqrt{2x-x^2}}$$

$$d\,(\mathrm{covers}^{-1} x) = -\frac{dx}{\sqrt{2x-x^2}}$$

If the base of the system of logarithms is constant, then:—

if $y = a^x$; $dy = a^x \log a dx$

$y = \epsilon^x$; $dy = \epsilon^x dx$.

METHOD FOR FINDING THE LENGTH OF CURVES.

When the equation of any curve is given in terms of x and y, y being the independent variable, the methods of Integral Calculus may be applied to find the length of the curve between any two points or between assumed *limits*, as follows:—

Denoting the desired length of the curve by **s**, and the two limits by **a** and **b**, the expression for the desired length is:

$$\int_{\mathbf{a}}^{\mathbf{b}} \left[\left(\frac{dx}{dy} \right)^2 + 1 \right]^{\frac{1}{2}} dy$$

The value of $\frac{dx}{dy}$ must be found in terms of y from the given equation of the curve.

EXAMPLE IV: The length of the curve of the parabola expressed by $y^2 = 2px$, between its vertex and the end of the latus rectum* is

$$\frac{p\sqrt{2}}{2} + \frac{p}{2} \log(1+\sqrt{2}).$$

EXAMPLE V: The length of the catenary curve is expressed by $y = \frac{a}{2}(\epsilon^{\frac{x}{a}} + \epsilon^{-\frac{x}{a}})$, between the points $(x, 0)$ and (x, y) is $\frac{a}{2}(\epsilon^{\frac{x}{a}} - \epsilon^{-\frac{x}{a}})$.

The preceding is useful as applying to the length of wires when suspended freely between two insulators, as in transmission lines.

The matter of the length of curves will be treated more at length in Vol. II. in relation to transmission lines.

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*The latus rectum is a line through the focus of the parabola, perpendicular to the X axis, when the parabola is located as indicated by the above equation, with its vertex at the origin of the coördinate axes, extending to the right from the Y axis, with the X axis an axis of symmetry.

INTEGRATION.

Integration is often defined as the *inverse of differentiation.* Many examples of inverse operations have already been given. A few *comparative* examples of inverse functions, will however be cited as follows:—

$x = y^2 + 1$	$y = \pm\sqrt{x-1}$
$x = \sin\phi$	$\phi =$ angle whose sine is $x = \text{arc} \sin x = \sin^{-1} x$
$y = \epsilon^x$	$x = \log_\epsilon y$.

The second member of each equation appearing in the first column above is the *inverse* function of the second member of the equation on the same line, appearing in the second column.

Applying the foregoing principle, integration would mean finding the value of a function from the differential of the function.

When a function is thus found, it is called the *integral* of the differential of the function. The operation is indicated by the integral sign $\int$.

The differential of the first equation given above is expressed by:— $dx = 2y dy$.

Applying rule 7, page 32, to y^2, and rule 1, page 32, to 1.

The *integral* of this differential would be expressed by:— $\int dx = \int 2y\, dy$.

This is $x = y^2 + C$, in which C denotes a constant which must always be added to the final integral expression. The numerical value of the constant of integration may be found by some mathematical process. Its value in the present case is 1.

The constant C may have an infinite number of values, and then the integral expression itself will have an infinite number of values. In such cases the expression is called the *indefinite integral* of the differential expression.

As a matter of fact integration is a process of summation, and was probably first designed with the object of summation in view. The symbol $\int$ denoting the process of integration is simply an old fashioned s, the first letter of the word "sum", originally written ſum.

One important use of integration, so far as the solution of engineering problems is concerned, is to furnish a method of finding the areas included by "curves", or between certain prescribed "limits" arbitrarily assumed.

When integrating an expression between prescribed "*limits*" the constant of integration is omitted.

The integral of a function or of an expression in this case is said to be a "*definite*" integral. There can be but *one* value of the integral.

The prescribed limits are of course *unequal* in value. The smaller is called the *inferior limit* and the larger the *superior limit.*

The inferior limit is designated by a figure or a symbol placed at the right of the *lower* end of the integration sign, while the superior limit is designated by a figure or symbol placed at the right of the *upper* end of the integration sign.

To designate that the function $dy = axdx$ is to be integrated between the limits of 2 and 5, it would be expressed by:—

$$\int_2^5 dy = a\int_2^5 xdx.$$

See pages 61, 62, 71, and 75, for other examples.

A definite integral is the limit of a *sum.*

It is unfortunate that there is no General Rule that may be applied for the process of integration. At present each problem requires specific or individual consideration, depending more or less upon the results of previous cases. The present method of integration is more of a "trial and error" method than is perhaps desirable.

To facilitate the application of the process of Calculus to engineering computations, a number of *typical* integral expressions have been obtained by mathematicians, to meet as many conditions as possible.

A few of these typical expressions have been arranged on page 41.

The chief difficulty encountered by the *engineer,* is in reducing the mathematical, or algebraical expression at hand, to one of the type forms. This process at times requires mental gymnastics; called by some, *mathematical artifices,* which are not easy or simple.

The following rules may be found useful as artifices:—

RULE I. **The integral of the algebraic sum of a number of *DIFFERENTIAL* expressions is equal to the algebraic sum of the *INTEGRALS* of the same expression; each considered separately.**

For example: differentiating the expression

$$\int dx + \int dy - \int dz$$

gives:—

$$dx + dy - dz. \quad \text{Hence:}$$

$$\int (dx + dy - dz) = \int dx + \int dy - \int dz$$

RULE II. A *CONSTANT* **factor may be written either before or after the integration sign.**

It is probably less confusing to write the constant factor (or factors) *before* the sign of integration.

This is shown by the following expressions:—

$$\int \mathbf{a}x\mathbf{d}x = \mathbf{a}\int x\mathrm{d}x$$

$$\int \mathbf{ab}y\mathbf{d}y = \mathbf{ab}\int y\mathbf{b}y.$$

A FEW TYPICAL INTEGRAL EXPRESSIONS.*

1. $\int \sin\theta \, d\theta = -\cos\theta + C.$ **2.** $\int \sin 2\theta \, d\theta = -\frac{\cos 2\theta}{2} + C.$

3. $\int \cos\theta \, d\theta = \sin\theta + C.$ **4.** $\int \cos\frac{\theta}{2} \, d\theta = 2\sin\frac{\theta}{2} + C.$ **

5. $\int \sin x \cos x \, dx = -\frac{1}{4}\cos 2x + C.$ **6.** $\int \cos mx \, dx = \frac{\sin mx}{m} + C.$

7. $\int \sin^2\theta \, d\theta = \frac{\theta}{2} - \frac{1}{4}\sin 2\theta + C.$ *Important.*

8. $\int \cos^2\theta \, d\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C.$

9. $\int \sin^2\theta \cos^2\theta \, d\theta = \frac{1}{2}\sin^2\theta + C.,$
10. $\int \tan\theta \, d\theta = \log.\sec\theta + C = -\log.\cos\theta + C.$
11. $\int \cot\theta \, d\theta = \log.\sin\theta + C.$

12. $\int \sec\theta \, d\theta = \int \frac{d\theta}{\cos\theta} + C = \log(\sec\theta + \tan\theta) + C = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + C.$

13. $\int \operatorname{cosec}\theta \, d\theta = \log\tan\frac{\theta}{2} + C.$

14. $\int x \, dx = \frac{x^2}{2} + C.$ **15.** $\int x^2 dx = \frac{x^3}{3} + C.$ **16.** $\int x^3 dx = \frac{x^4}{4} + C,$ etc.

17. $\int \frac{2x \, dx}{x^2+1} = \log(x^2+1) + C = \log(x^2+1) + \log C = \log[C(x^2+1)].$

18. $\int \sqrt{a^2 - x^2} \, dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \operatorname{arc}\sin\frac{x}{a}\right) + C.$

If the *base* of the system of logarithms is a *constant*, then:—

19. $\int a^x dx = \frac{a^x}{\log a} + C.$ **20.** $\int x \epsilon^{mx} dx = \frac{1}{m^2}\epsilon^{mx}(mx - 1) + C.$

21. $\int \epsilon^x dx = \epsilon^x + C.$

22. $\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x \, dx + C.$

23. $\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x \, dx + C.$

24. $\int \sin mx \sin nx = -\frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C.$

---

*A very valuable collection of type forms is given in Granville's "Differential and Integral Calculus", published by Ginn & Co.

**From the general expression; $\int \cos mx \, dx = -\frac{\sin mx}{m} + C.$

DEFINITIONS.

ALTERNATING QUANTITIES.

An alternating quantity may be defined as a quantity which changes from positive to negative maxima values during equal and regular recurring intervals. Between these maxima values the instantaneous values or the values at different instants, may have very different values. Whatever the variation in the instantaneous values may be, it always takes place according to some definite *law*.

There are three important alternating quantities met with in Electrical Engineering; **alternating-pressures**, or E. M. F.s, **alternating-currents,** and **alternating-magnetic-fields.**

According to the definition, *each* of the three quantities **vary in** both **magnitude** and **direction** during equal intervals of time, and may be alluded to as a *wave*.

It makes no difference what the *shape* of the alternating wave may be, since it is a periodic function of time (time being the independent variable, while the variation in the instantaneous values of the alternating quantity is the dependent variable), it is always made up of a sum of a number of simple *sine-waves*; the component sine-waves may have different maxima values, different frequencies, and different starting points. This fact is known as Fourier's Theorem*, which should be carefully studied.

All the alternating quantities mentioned are single valued functions; that is for one assumed value of the independent variable there is one and only one value of the dependent variable. This may be better appreciated by consulting any of the equations expressing an alternating-pressure or current, and substituting therein any assumed value for the independent variable; such as time in seconds or in fractions of one second.

When considering a *"direct-current"*, the current is assumed to have a certain *direction*, and while the magnitude or the amount of the current may vary (either increase or decrease) its *direction* remains unchanged.

Definition of Alternating–Current.

An *alternating-current* of electricity, however, is one which changes its *direction* of action, as well as its magnitude, always according to some definite law of regularity; which always provides that the *direction* of action must change at certain definite intervals. The *regularity* of the change in both magnitude and direction, is a most important characteristic of an alternating-current.

---

* Byerly, *Fourier's Series and Spherical Harmonies*, page 30.

An alternating-current, as defined, may be represented by a curve, as shown in figure 12, or in figure 30, page 109 in which all distances measured upward from OX are considered as positive (+); while distances measured downward from OX are negative (−). If a current is considered to start from O and to increase according to some law in a positive direction, it reaches a certain maximum value, denoted by + m, at which point it begins to decrease according to a definite law until point d is reached, when its value is again zero. From this point it continues to increase in the opposite or *negative direction*, until −m is reached, when it begins to decrease, numerically, continuing until point h is reached, when its value again becomes zero.

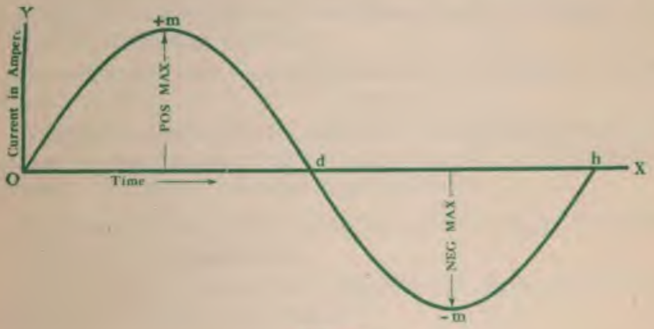

FIG. 12.

The continuous repetition of such changes constitutes an alternating-current.

While it is *possible* for an alternating-pressure to have the form of a sine-curve, such is seldom the case in ordinary practice. For some purposes a sine-curve shape would be desirable, while for some other important results, forms far from a sine-wave shape would be more desirable. Since pressure, or E. M. F., is the cause of the so-called electric current, an alternating-pressure applied to the terminals of any circuit will produce an *alternating-current* in that circuit. It does not follow, however, that a *sine-curve* pressure when applied to the terminals of a circuit will produce in that circuit an alternating-current having a sine-curve form. While the resulting current is an *alternating*-current, it may be distorted from a sine-curve shape. The causes that may distort a wave-form are numerous; a common one is the variation in the magnetic permeability of the medium surrounding the circuit, at different flux densities.

Another is the variation in magnetic flux density of the field through which an inductor is moving with uniform motion. Also variable motion of an inductor in a uniform field, will produce a non sine-wave of pressure.

If on the other hand the alternating-pressure applied to any circuit has a shape that is not a sine-curve shape, the resulting current, although it is an alternating-current, will not usually have the same shape as the applied pressure.

FREQUENCY.

When a current has passed from zero to a positive maximum, to zero and through a negative maximum back again to zero, it is said to have completed one *cycle*. The curve in figure 12, page 43, denotes one cycle.

The interval of time during which an alternating-current completes one cycle is, in practice, a fraction of a second; from $\frac{1}{125}$ to $\frac{1}{15}$ of a second; is called its *period*, or its *periodic-time*, and is denoted by the letter **T**. The number of complete cycles of changes in one second is termed the *frequency*, and is denoted by the letter f.

The relation between the periodic-time and the frequency is expressed by $T = \frac{1}{f}$.

In figure 12, time in seconds (or fraction of a second) is plotted horizontally, and the distance from O to h might denote $\frac{1}{125}$ or $\frac{1}{15}$ of one second, as desired. In this case **T** is denoted by the line Oh.

The character $\sim$, called the tylde, is often used to denote *cycles*; **25** $\sim$ means 25 cycles per second.

The alternating-current curve illustrated in figure 12, page 43 is a "*sine-curve*", the equation for which is:

$$i = A \sin \theta = A \sin (2\pi ft.) \qquad (\theta = 2\pi ft).$$

In this equation i denotes an instantaneous value which is constantly changing; A denotes a constant, and θ denotes a varying angle; considered as made up of several factors; some constant, 2π and f, and one, (t), a variable.

Curves alluded to or discussed in this book are referred to rectangular co-ordinate axes; the vertical axis, or the axis of ordinates being designated as the **Y** axis, and the horizontal axis, or the axis of abscissae being designated as the **X** axis. These two perpendicular axes or datum lines are assumed to intersect at a point designated by the letter O. The point of intersection O is sometimes called the *origin* of this system of co-ordinates.

If a curve is a symmetrical one, such as a circle, an ellipse, a parabola, or a triangle (isosceles), the curve is usually located with one of its axes (axis of symmetry) coincident with one of the reference axes of the system.

In many cases, such curves as the circle and the ellipse, are located with their *centers* at the origin O of the coordinate axes.

Such curves are designated by equations expressed in terms of x and y, which denote the *variable* quantities. For example, x may denote a *variable time*, a *variable angle* in degrees or radians, or a *variable velocity*; while y may denote a varying E. M. F. or pressure, a varying current, or a varying magnetic field.

The equation of a circle when referred to rectangular coordinates with its center at the origin O is:

$y=\sqrt{r^2-x^2}$; of an ellipse under the same conditions is:

$y=\pm\frac{a}{b}\sqrt{b^2-x^2}$, if b denotes its major axis, a denotes its minor axis, and the ellipse is located as shown in figure 24, page 80. (See page 70 and page 81.)

If a curve is a *time-function* curve such as a sine-curve, then the abscissae or x values denote the *independent variable*, time; (being denoted by t, and usually expressed in seconds or fractions of a second)or some quantity depending directly upon the time. In this case the ordinates or y values denote a *dependent variable*, such as an alternating-pressure, an alternating-current or an alternating-field.

Then $y=\mathbf{A}\sin x$ might be written as:

$e=\mathrm{E}_{\max}\sin(2\pi ft)$, to denote a sine-pressure;

as; $i=\mathrm{I}_{\max}\sin(2\pi ft)$, to denote a sine-current;

or as; $\phi=\Phi_{\max}\sin(2\pi ft)$, to denote a sine-field.

Whenever a curve is designated in terms of x and y, a *general* variable quantity is denoted, and when these letters are replaced by other letters, a *specific* variable quantity is denoted.

A parabola, when its axis is coincident with the **Y** axis and its vertex is pointing upward, is designated by:

$$y=\frac{\mathrm{m}-x^2}{\mathrm{p}}.$$ See figure 25, page 84.

while a parabolic pressure, or E. M. F., similarly located, would be expressed by:

$$e=\frac{\mathrm{m}-t^2}{\mathrm{p}}:$$

in which equation, e denotes the instantaneous value of the variable pressure, at the end of an interval of time (denoted by t) after a positive maximum* value of the same pressure. The constants m and p may be evaluated if *two* corresponding values of e and t are known.

---

*Note well that in this case time is not reckoned from the beginning of a positive lobe, but from the *middle* of a positive lobe.

PRODUCTION OF ELECTRO-MOTIVE FORCE.

In order to study the production of an *induced* electro-motive force, having the shape of a *sine-curve*, by a so-called alternating-current generator, or as it is commonly called, an *alternator*, let figure 13 represent a *uniform* magnetic field, having a total strength of Φ lines of force, ($\Phi =$ area of the coil times the lines per unit area) and a *direction* from pole N to pole S, in which magnetic field a rectangular coil of copper wire denoted by c d e f, is so arranged that it may be turned about an axis denoted by AX, which passes through the center of the field, and is perpendicular to the magnetic lines of force of the field.

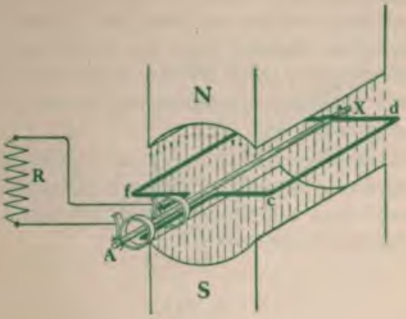

FIG. 13.

The wire of the coil is insulated from the shaft, and has one end attached to one collector ring, and the other end attached to the other collector ring as indicated in the figure. The collector rings are supposed to be insulated from each other and from the shaft. Two brushes make sliding contact; one with each ring and thus make electrical connection between the rotating coil and a resistance denoted by R; which may be called the external circuit.

The distance from the axis AX to either side of the coil is considered to be unity.

Assume the initial or starting position of the coil to be when its plane is horizontal, or perpendicular to the direction of the magnetic field, and that the direction of rotation of the coil shall always be *counter-clockwise*.

Suppose the coil to rotate at a constant or uniform speed of n revolutions per *second*. The time of one complete revolution will then be $\frac{1}{n} = \mathbf{T}$.

Here **T** denotes the period or the periodic time as mentioned on page 45.

In the case of a two-pole field, $\frac{1}{n}=\frac{1}{f}=T$. That is, there are as many complete cycles of induced pressure, per second as there are complete revolutions per second.

In the assumed horizontal initial position, the number of lines of force included by (or passing through) the coil, is a maximum.

Since the coil might be rotated through some distance either side of the horizontal position, without producing but a slight variation in the number of its including magnetic lines, the *rate of cutting* magnetic lines while moving through a very small distance in the region of the initial position, is very small, and *when* actually *moving through the* assumed *initial position, the rate of cutting is zero.*

Therefore the electromotive force induced in the coil, due to the rate of cutting lines, or due to the rate of change in the included magnetic flux, must also be zero, when the coil is passing through a horizontal plane. This means zero current in the external circuit.

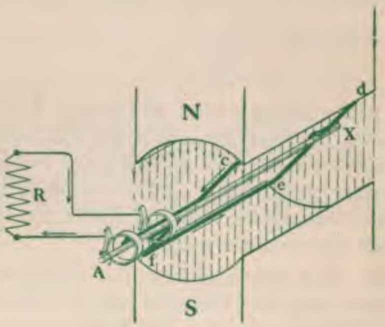

FIG. 14.

Next suppose that the coil in its motion has completed $\frac{1}{8}$ of one revolution and is in a position as indicated in figure 14. The effective number of magnetic lines now passing through the coil in its new position will be $\Phi \cos 45°$.

Further; when the coil has completed $\frac{1}{4}$ of a complete revolution and is in a position denoted by figure 15, the number of included lines will be *zero*, but the rate of cutting the lines will now be a *maximum.*

The *direction* of the induced pressure may be ascertained for any position of the coil by applying the *right-hand rule,* as given in "Generator and Motor Examples". The directions of the induced pressures and the resulting currents are indicated by arrows in figures 13, 14, 15, 17 and 18.

Since the induced pressure is a *variable* one, it will be well to consider an expression for the induced pressure that will enable its *numerical* value at *any* instant to be computed.

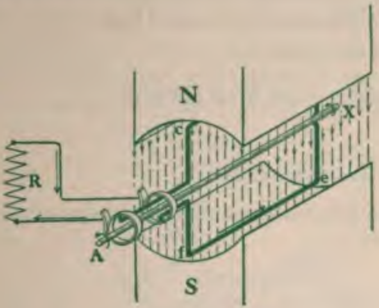

FIG. 15.

Any induced electromotive force, denoted by e, depending in value upon the "*rate of change*" in magnetic flux, may be expressed, in terms of Calculus, by:

$$e = -\frac{d\phi}{dt}.$$

The negative sign in this case simply denotes that the induced pressure is in such a direction as to oppose the motion of the coil; according to the law of the *conservation of energy.*

The law governing the variation in the flux ϕ, will therefore determine the instantaneous numerical values of the induced pressure e. In other words; if the magnetic field is *not* uniform, even though the motion of an inductor is uniform, the induced pressure will not undergo a uniform variation.

If the field as shown in figure 13 were *not* uniform, with uniform rotational motion of the coil as shown, a *non* sine-wave of induced pressure would result.

To make the problem a *general* one, suppose the coil in figure 13 has turned through any angle whatsoever, that may be designated by α.

The *time* required to do this may be designated by t.

If angle be measured in *radians*, one complete revolution will be 2π (*radians*). If the coil rotates n revolutions per *second*, the angle turned through in one second will be $2\pi n$ (*radians*). Therefore the angle turned through in any time t, will be $2\pi nt$.

Then:

$$\alpha = (2\pi nt) \qquad \text{(t may be a small fraction of a second.)}$$

The magnetic flux through the coil when it is at an angle α with the assumed initial position may be expressed by: (see figure 15a.)

$$\phi = \Phi \cos (2\pi nt)$$

but $e = -\frac{d\phi}{dt}$, and $d\phi = -2\pi n \, \Phi \sin (2\pi nt) dt$

Therefore:

$$e = +2\pi n \, \Phi \sin (\pi 2nt)$$

This shows the induced pressure e to be a sine-wave pressure, of the form;

$$e = \mathrm{E}_{\max} \sin \alpha;$$
$$\mathrm{E}_{\max} \sin (2\pi nt) = \mathrm{E}_{\max} \sin (2\pi nt);$$
$$\mathrm{E}_{\max} \sin (2\pi ft) = \mathrm{E}_{\max} \sin (2\pi ft); \qquad \text{(in the assumed case $n = f$.)}$$
$$= 2\pi n \, \Phi \sin (2\pi ft). \quad (1)$$

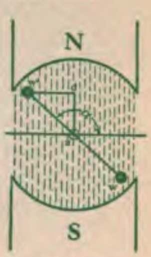

FIG. 15a.

Such a curve is shown graphically in figure 16, also in Plate I., page 1, in which OA′ denotes the *position* of the rotating coil at a certain instant, (when $2\pi ft = \alpha$), while the *length* of OA′ is *proportional* to the *maximum* value (denoted by $\mathrm{E}_{\max}$), that the induced pressure ever assumes.

When the coil is in the assumed initial position, angle $\alpha = 2\pi nt = 0$, and the induced pressure $e = 0$. (sine of 0° = 0.)

The so-called "*radius vector*" OA′ will then occupy a position OA, along the X axis.

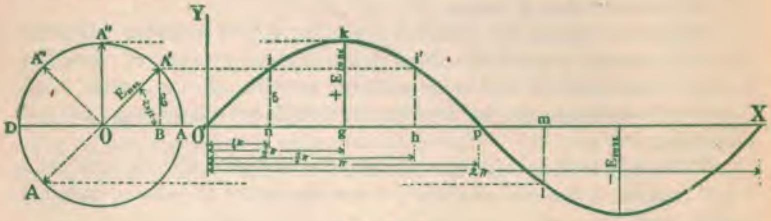

FIG. 16.

When the coil is parallel with the magnetic lines of the field, angle $\alpha = 2\pi ft = 90°$; but since the sine of 90° is equal to 1, then; $\mathrm{E}_{\max} = 2\pi f \, \Phi$; being the maximum value it can ever have, and which is denoted by OA″, figure 16.

There is one instant during each half revolution, when the instantaneous value of the induced pressure is numerically equal to the maximum value.

Then:

$$e = E_{max} = -E_{max}$$

As the coil continues to rotate from the 90° or $\frac{\pi}{2}$ position, (see figure 17), e is *decreasing* in value, and when $\alpha = 2\pi ft = 180°$, e is again $= 0$, and the coil is again in a horizontal position; but the side c d is now where e f was in the assumed initial position, and relative to the *coil* the direction of the lines of force through it is just opposite to the direction initially.

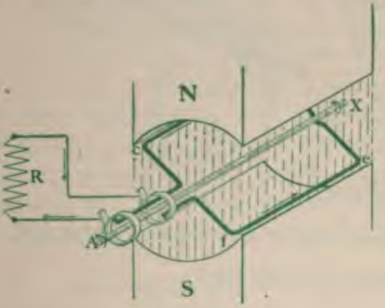

FIG. 17.

Therefore during the next quarter revolution the induced pressure (and resulting current) is *increasing* (numerically) in the opposite direction, (see figure 18) to that during the first quarter revolution; similarly during the *fourth*, quarter revolution, the pressure is *decreasing* in the opposite direction to that during the *second*, quarter revolution.

By assuming various values of angle from 0° to 360°, and substituting their natural sines in equation (1), page 50, a complete cycle of induced pressure may be plotted.

If instead of only *one* revolving rectangular coil, there were **N** coils, or one coil consisting of **N** turns of wire, wound close together and occupying only a small fraction of the whole circumference then the induced pressure will be **N** times as great; or $e = N 2\pi f \Phi \sin(2\pi ft)$.

Since the direction of the pressure was considered as positive during the first half revolution, the various values of pressure were plotted above the line OX; or above the X axis, in figure 16. Since values during the second half revolution were in the *opposite* direction, they were plotted downward, *below* the X axis, and designated as *negative*.

It is evident from the foregoing explanation that the length of the radius vector **OA′** denotes the maximum value of the induced pressure, and by its angular relation with the horizontal datum line OX, denotes the angular position of the rotating coil indicated in figures 13, 14, 15, 17 and 18.

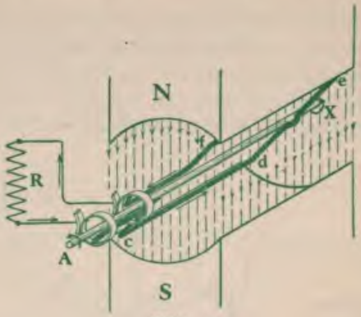

FIG. 18.

It may be said that:

$OA' = E_{max}$, and by trigonometry, in triangle **OA′B**, figure 16,

$A'B = e = OA' \sin (2\pi f t)$.

If the distance $O'p$, along OX is equal to the length of the semicircular arc **AA′A″D**, and pX is equal to the same length, $\frac{1}{4}$ the distance from **O′** to p, (**O′**n) is $\frac{\pi}{4}$ (radians) or 45°. If at this point a perpendicular, n i is erected, equal in length, (or proportional to);

$$\text{sine of } 45^\circ \times E_{max} = \frac{1}{\sqrt{2}} \times E_{max} = 0.707 \ E_{max},$$

it will be one ordinate of a so-called sine-curve of pressure.

The sine-curve shown in figure 16 is a *true* sine-curve so to speak; meaning that the maximum ordinate of the curve is equal in length to the radius of the generating circle **AA′A″D**. This might be stated in another way by saying that **A** in the general sine-curve equation, $y = A \sin x$ is **1**; or $E_{max} = 1$ in the case under consideration.

If the radius of the generating circle is *unity*, say 1 inch, then the circumference of the circle is 3.14159 inches, and the horizontal distance O′X is also 3.14159 inches.

If E_{max} had been 100 instead of 1, the *base* of the sine curve would have been 3.14159 or 2π, but the maximum ordinate would have been 100 inches instead of 1 inch. It is customary to consider alternating-quantities as having a base 2π in length; that is, based on *unit circle*; but in representing such quantities by means of graphs or curves, the proportionate relations are not adhered to. If such were the case, a curve containing the 100 volt (maximum) pressure just alluded to would have a base 3.14159 (2π) inches long, a positive maximum 100 inches long and a negative maximum 100 inches long; requiring a sheet 4 inches wide and over 200 inches in length, if plotted to scale.

When **OA′** in figure 16, page 50, see also figure 15a, page 50, has an angular position 135° (or $\frac{3}{4}\pi$), the e ordinate will be the same in length as for the 45° position, but located at hi'.

When **OA′** has a position 225°, (or $\frac{5}{4}\pi$), the instantaneous value of e is denoted by lm. $lm = hi'$, but the sine of 225° being *negative*, lm is drawn *downward* from the datum line.

While **OA′** and likewise the rotating coil, makes one complete rotation, the induced pressure e goes through one complete cycle of changes.

The curve **O′***iplX*, denotes a complete cycle; denoted by $\sim$.

One revolution of the armature coil in a two-pole alternator, will produce one complete cycle of alternating pressure, and this alternating pressure acting on a circuit will produce one complete cycle of alternating-current.

Since in the case under consideration $n = f$ the pressure may be expressed by:

$e = E_{max} \sin (2\pi ft)$, and the current by:

$i = I_{max} \sin (2\pi ft)$; provided it is a sine-wave current and in phase with the induced pressure.

If the coil described, rotates 25 times in one second a 25 $\sim$ pressure will be produced, and if a closed circuit exists, a 25 $\sim$ current will result.

Should the simple alternator described, be provided with *four* poles; two **N** and two **S** with like poles opposite each other, the same speed of rotation would produce *twice* the number (50 $\sim$) of complete cycles; or the coil might be operated at one-half the speed and produce 25 $\sim$. This does not mean the numerical value *in volts* of the 50 $\sim$ pressure and the 25 $\sim$ pressure will be the same. The intensity and frequency of an alternating-pressure are quite independent of each other.

An equation showing the relation of frequency, number of poles and speed may be given as follows:

Let $f = pn$; in which f denotes the frequency, or the number of complete cycles per second; p the *number of pairs* of poles, and n the number of revolutions per *second*.

Since the speed of machines is usually obtained and expressed in revolutions per minute, letting V denote the revolutions per minute, the above equation becomes:

$$f = p\frac{V}{60}$$

$$n = \frac{V}{60}$$

If p denotes the **number of poles,** then:

$$f = \frac{p}{2} \times \frac{V}{60} = \frac{1}{120} p\, V.$$

This fundamental equation applies to any alternator with a revolving armature, and also to *synchronous* motors; when it is desired to compute their armature speeds.

Practical Application.

Since the revolutions per second, and the frequency of the current of a two-pole alternator are the same numerically, a small two-pole alternator may be employed as a *speed indicator.* If a small "magneto" or telephone bell ringing generator, be electrically connected with a frequency meter, such as a Frahm vibrating reed instrument, the apparatus becomes a very reliable speed indicator. The frequency indicated by the frequency meter being the speed of the magneto armature, *per second.* The magneto armature may be belted to, or direct-coupled with, the shaft whose speed is desired.

EXAMPLE 1: Find the speed, in revolutions per minute, of a 24 pole alternator, to yield a pressure at 60 cycles.

The general equation for this case is

$$f=\frac{\mathbf{P}}{2}\,\frac{\mathbf{V}}{60}\ ;\ \text{in which}$$

f denotes cycles per second;
P denotes the number of poles;
V denotes the speed in revolutions per *minute*.

DATA.
$f=50$ cycles
$\mathrm{P}=24$ poles

Solving the above equation for **V**, after making the proper numerical substitutions gives:

$$\mathbf{V}=\frac{2\times60\times60}{24}=\mathbf{300\ revolutions\ per\ minute.}\quad\mathbf{Answer.}$$

PROBLEM 1: Find the required speed, in revolutions per minute, of a 4 pole alternator, to produce a pressure having a frequency of 60 cycles per second. **1800 revolutions per minute. Answer.**

Problem 1a: Find the speed, in revolutions per minute, of a 24 pole alternator to produce a pressure having a frequency of 25 cycles per second.

Problem 1b: Find the speed, in revolutions per minute, of a 4 pole alternator to generate a pressure having a frequency of 25 cycles per second.

INDUCED E. M. F's.

EXAMPLE 2: A straight conductor 1 meter long is displaced parallel with its initial position and at right angles to a uniform magnetic field at a rate of 100 meters per second. If the field strength is 100 C. G. S. units, what E. M. F. is generated in the conductor?

The general equation is:
$e^1 = l\mathrm{Hv}$ abvolts.
$e=\frac{l\mathrm{Hv}}{10^8}$ **volts.**

DATA.
$l=1$ meter $=100$ centimeters.
$\mathrm{H}=100$ C. G. S. units. (gausses)
$\mathrm{v}=10000$ centimeters per second.

Substituting the given numerical values gives:

$$e=\frac{100\times100\times10000}{10^8}=\frac{100000000}{100000000}=\mathbf{1\ volt.}\quad\mathbf{Answer.}$$

This not an alternating-pressure.

PROBLEM 2: A straight wire 1 foot long is displaced parallel with its initial position and at right angles to the direction of the lines of force in a uniform magnetic field at a rate of 100 feet per second. If the field strength is 1000 gausses, find the induced pressure, in volts, between the ends of the moving wire. **0.929 volt. Answer.**

Problem 2a: A straight wire 39.37 inches long is moved parallel with its initial position, and perpendicular to a uniform magnetic field at a rate of 32.8 feet per second. If the field strength is 100 C. G. S units, (gausses) find the E. M. F. induced in the given length of wire.

EXAMPLE 3: A straight conductor 1 meter long moves always parallel with its initial position in simple harmonic motion with an amplitude of 1 centimeter, at right angles to a uniform magnetic field having a strength of 1000 gausses. If the periodic time* is $\frac{1}{1000}$ second, what maximum E. M. F. is induced in the conductor?

The induced pressure at any position of the moving wire is expressed by:

$$e = \frac{Hv}{10^8} \text{ volts}$$

DATA.

$l = 100$ cms.

$\frac{1}{f} = T = \frac{1}{1000}$ second.

$H = 1000$ gausses.

The length l, of the wire is constant; H is a constant; v is a variable. The velocity of the moving wire *varies*, being zero at the extreme positions of the wire, and a maximum when the moving wire is passing through a position half way between its two extreme positions. The question is then, to find the velocity of the moving wire when passing through its median position.

The *maximum* velocity of the moving wire is the same as it would have if moving with uniform motion in a circle whose radius is 1 centimeter, at 1000 revolutions *per second*. The value of v in the above equation thus becomes:

$$v = 2\pi \times 1000 \text{ centimeters per second.}$$

Substituting the proper numerical values the induced pressure becomes:

$$e = \frac{100 \times 1000 \times 1000 \times 2\pi}{100{,}000{,}000} = 2\pi \text{ volts} = 6.28 \text{ Volts.} \quad \textbf{Answer.}$$

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*See page 45.

PROBLEM 3: A straight conductor 1 foot long, is arranged to always move parallel with its initial position, in simple harmonic motion, with an amplitude of 1 foot, at right angles to a uniform field having a strength of 1000 gausses. If the periodic time of the moving wire is $\frac{1}{10}$ of a second, find the value of the **maximum** *induced pressure.*

v = 1915.12 *cms. per second.* **E_{max} = 0.583 volts. Answer.**

Problem 3a: A straight conductor 2 feet long, is arranged to always move parallel with its initial position, in simple harmonic motion, with an amplitude of 1 foot, at right angles to a uniform magnetic field having a strength of 1000 gausses. If the periodic time of the moving wire is $\frac{1}{5}$ of a second, find the value of the maximum induced pressure.

INDUCED A. C. PRESSURES.

EXAMPLE 4: A conductor bent into the form of a rectangle whose longer dimension is 100 centimeters and whose shorter dimension is 30 centimeters is arranged to be turned about an axis which is parallel with the longer dimension, as indicated in figure 13, page 47, with uniform angular velocity of 10 revolutions per second in a uniform magnetic field having a strength of 800 gausses; so that in one position of the rectangle the magnetic lines of force pass normally or perpendicularly through it.

Find the value, in volts, of the maximum E. M. F. induced in the rectangular circuit.

The maximum induced pressure is expressed by:

$$e^1 = HlV \text{ (abvolts)}$$

$$= \frac{800 \times 200 \times 2\pi \times 15 \times 10}{10^8} \text{ volts}$$

$\doteq$ **1.509 volt. Answer.**

DATA.

$H = 800$ gausses.

$l = 2 \times 100 = 200$ cms.

$r = 15$ cms.

$n = 10$ revs. per second.

$V = 2\pi rn$.

$\pi = 3.14159$.

1 volt $= 10^8$ abvolts.

PROBLEM 4: If there are 100 **turns** *of wire in the rectangular circuit mentioned in* EXAMPLE 4, *find the value of the induced pressure, if all other conditions remain as there stated.* **150.9 volts. Answer.**

Problem 4a: Suppose the axis of revolution passes parallel with the shorter dimension of the coil mentioned in EXAMPLE 4. Find the value of the induced pressure, in volts, if all other conditions remain as stated in the example. Compare result with EXAMPLE 4.

EXAMPLE 5: If electricity travels along a wire circuit with a velocity of 188,000 miles per second, find the wave-length, in miles, of a sine-wave current having a frequency of 60 cycles per second.

DATA.

$f = 60$ cycles

$V = 188{,}000$ miles

λ denotes the length of one wave or cycle, in miles.

—O—

If the current travels 188,000 miles in one second and there are 60 complete cycles per second, the length of one cycle must be:

$\lambda = \frac{V}{f}$ or $\lambda = \frac{188{,}000}{60} = 3133\frac{1}{3}$ **miles.** **Answer.**

PROBLEM 5: If electricity travels a circuit with a velocity of 188,000 miles per second, find the wave-length, in miles, of a sine-wave current having a frequency of 25 cycles per second.* **7520 Miles. Answer.**

Problem 5a: If electricity travels through space at a speed of 188,000 miles per second, find the frequency necessary to produce a wave-length of 163.68 feet.

Problem 5b: The wave length of the radiation in the solar spectrum at the **D** line is $\frac{1}{45000}$ inch. What must be the frequency of an electrical discharge to produce such a wave-length?

PROBLEM 5c: If a wireless telegraph impulse travels at the rate of 188,000 miles per second, how long will it take for an impulse to arrive at the sun after leaving an antenna on the earth? The sun is said to be 91,000,000 miles from the earth.

478.7 seconds; or nearly 8 minutes. Answer.

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*The velocity of light and the velocity of electricity is assumed to be the same; 188,000 miles per second.

EXAMPLE 6: Find the instantaneous value, in volts, of the pressure, at a time 5.003 seconds after the beginning of a cycle, between the terminals of an alternator producing a sine-curve pressure, whose effective value is 707.1 volts, having a frequency of 25 cycles per second.

The equation applying is:

$$e = \mathrm{E}_{\max} \sin(2\pi f t).$$

Making the proper numerical substitutions gives:

$$e = \mathbf{1000} \sin(2\pi \times 25 \times 5.003)$$
$$= 1000 \sin 250.15\pi = 1000 \sin(0.15\pi)$$
$$= 1000 \sin 27^\circ = 1000 \times 0.45399 =$$

453.99 volts. Answer.

DATA.

$\mathrm{E}_{\mathrm{eff}} = 707.1$ volts.
$\mathrm{E}_{\max} = 707.1 \times 1.414$
$= 1000$ volts.
$f = 25$ cycles.
$t = 5.003$ seconds.

It should be considered that an angle may be *more* than 360° or more than 2π *radians*. In most text-books dealing with alternating-currents, angle is measured in *radians*. Tables of trigonometric functions are however usually given with reference to angles measured in *degrees*.

1 radian $= \frac{360}{2\pi}$ or $\frac{180}{\pi} = 57.2957^\circ = 57^\circ 18'$. (See page 7.)

If an angle is 360°, plus more degrees, its sine and cosine is the same in value as for the given angle *minus* the 360°. It is therefore evident that so far as evaluating the trigonometrical functions of an angle of a *large number* (larger than 360) of degrees is concerned, all *even* values of π, or of 360°, may be rejected.

For **example,** *what is the sine of* **40.2π?**

$\pi = 180^\circ$ and $40.2\,\pi = 40.2 \times 180^\circ = 7236^\circ$.

The question now is, what is the sine of 7236°?

To produce this angle, a radius vector, or generating line, must rotate $\frac{7236}{360} = 20.1$ times. The sine of 360° is 0; the sine of $20 \times 360^\circ$ is 0. The sine of 7236° is therefore the same as the sine of $\frac{1}{10}$ of 360°, or of $\frac{2}{10}\pi$, which is the sine of $36^\circ = \mathbf{0.58779}$. *The sine of* $40.2\,\pi = \mathbf{0.58779}$ **Answer.**

Again: *What is the sine of* **$41.2\,\pi$?**

This is the same as the sine of 7416° or of $\frac{7416}{360}$; or of $1.2\pi = \frac{12}{10} \times 180^\circ = 12 \times 18^\circ = 216^\circ$. Sine of $216^\circ = -\mathbf{0.58779}$. **Answer.**

Further: *What is the sine of* **$43.6\,\pi$?**

This is the same as the sine of 7848°; or of $\frac{7848}{360}$; or of $1.6\pi = 288^\circ$. The sine of $288^\circ = -\mathbf{0.95106}$. **Answer.**

The sine of $2.1\pi = \sin 378^\circ = \sin(378^\circ - 360^\circ) = \sin 18^\circ = +0.30902$ (even multiples of π discarded).

The sine of $1.1\pi = \sin 198^\circ = -\sin(198^\circ - 180^\circ) = -\sin 18^\circ = -0.30902$ (*odd* multiples of π not to be discarded).

This illustrates the necessity of considering carefully the *sign* of the trigonometric functions in solving problems relating to alternating-currents.

PROBLEM 6: Find the **instantaneous value,** *in volts, of the pressure between the terminals of an alternator that produces a sine-wave pressure having a frequency of 60 cycles per second, and an effective value of 707.1 volts, at a time 5.003 seconds after the completion of a cycle.*

904.6 volts. Answer.

Compare this result with that in EXAMPLE 6.

Problem 6a: Assuming that a magnetic field varies according to the sine-curve law, find the instantaneous value of the field, in maxwells, at a time 5.003 seconds after the flux is zero and starting to increase in the positive direction, if the effective value of the flux is 7.07 megamaxwells, and the frequency 60 ∼.

$\phi = \Phi_{max} \sin (2\pi f t.)$ ("mega" means one million.)

EXAMPLE 7: Find the instantaneous value of the current, in amperes, at a time 5.003 seconds after the completion of a cycle, in a circuit in which the effective value of the current is 70.7 amperes the frequency 25 cycles, and the form of the current assumed to be a sine-curve.

The general equation for the above conditions is:

$$i = I_{max} \sin (2\pi f t).$$

Making the required numerical substitutions gives:

$$i = 100 \sin (2 \times \pi \times 25 \times 5.003.)$$
$$= 100 \sin 250.15\,\pi = 100 \sin .15\,\pi.$$
$$= 100 \sin 27° = 100 \times 0.45399 = 45.39 \text{ amperes.}$$

Answer.

DATA.

$I_{eff} = 70.7$ amp.
$I_{max} = 1.414 \times 70.7$
$= 100$ amperes.
$f = 25$ cycles.
$t = 5.003$ secs.
$\pi = 180°$.

———O———

The application of fundamental principles in this case may be appreciated by consulting pages 45 to 46 and figures 13, 14, 15, and 18, pages 47, 48, 49 and 52.

PROBLEM 7: Find the instantaneous value of the current, in amperes, at a time 5.003 seconds after the completion of a cycle in an alternating-current circuit, in which the effective value of the current is 70.7 amperes, the frequency 60 cycles, and the current a sine-wave. **90.48 amperes. Answer.**

Problem 7a: Find the instantaneous value of the magnetic field, in maxwells, at a time 5.003 seconds after the completion of a cycle, in a field in which the effective value of the flux is $\frac{1}{10}$ megamaxwell, and the frequency 25 cycles; assuming the field varies according to the sine-curve law.

AVERAGE VALUE OF SINE-CURVE.

EXAMPLE 8: The maximum value of an alternating-current having the form of a sine-curve is 100 amperes; what is its average value for a half period, from 0 to π?

DATA.

$I_{max} = 100$ amperes.

—O—

The expression for a sine-wave current is:

$$i = I_{max} \sin (2\pi f t).$$

The average value of the current will be the area of the lobe OAB indicated by the unshaded portion in figure 19, divided by the base $\frac{1}{2f}$, as shown. The time from O to B $= \frac{1}{2f}$.

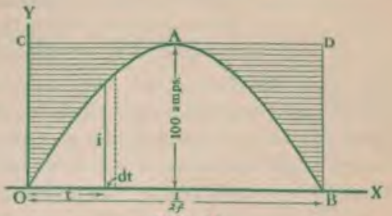

FIG. 19.

The area of the positive lobe of the current curve having a sine-wave form is expressed by:

$$\int_0^{\frac{1}{2f}} i\,dt = I_{max} \int_0^{\frac{1}{2f}} \sin (2\pi f t)\,dt$$

$$= I_{max} \left[\frac{-\cos (2\pi f t)}{2\pi f} \right]_0^{\frac{1}{2f}} = \frac{I_{max}}{2\pi f} \left[+1 - (-1) \right]^{*}$$

$$= \frac{2}{2\pi f} I_{max}$$

Average value $= \frac{2f}{1} \times \frac{2}{2\pi f} I_{max} = \frac{2}{\pi} I_{max} = .6366 \times 100 =$ **63.66 amp's. Ans.**

The area of the including rectangle OCDB, figure 19, of one lobe of the sine-curve in terms of the maximum ordinate $\mathbf{I_{max}}$ and the base $\frac{1}{2f}$ is expressed by:

$$\frac{1}{2f} \times I_{max}$$ (base multiplied by the altitude.)

But the area of one lobe of the sine-curve as found, is expressed by:

$\frac{I_{max}}{\pi f}$. This is true for *any* sine-curve.

Hence the ratio of the two areas is expressed by:

$$\frac{\text{area of sine-curve}}{\text{area of including rectangle}} = \frac{2 I_{max}}{\pi I_{max}} = \frac{2}{\pi};$$

$$= \frac{2}{3.14159} = \frac{1.273}{2} = 0.6366;$$

or the area of one lobe of a sine-curve = .636 times the area of its including rectangle.

---

* When $t = \frac{1}{2f}$, $\cos\pi = -1$, and $-\cos\pi = +1$. When $t = 0$, $\cos 2\pi f t = 1$, and $-\cos 2\pi f t = -1$.

Subtracting the inferior limit from the superior limit gives $+1 - (-1)$ or 2.

$\frac{1}{2f}$ *represents* $180°$ *or* $\frac{1}{2}$ *a revolution* of an ideal armature as shown in figure 13, page 47.
The time of $\frac{1}{2}$ *a revolution* $= \frac{1}{2f}$

PROBLEM 8: Find the times, in fractions of a second, after the beginning of a positive lobe, during one complete cycle, at which a 60 cycle sine-wave alternating-pressure assumes its **average** *value.*

1st at end of .0018302 second.
2nd at end of .0065031 second.
3rd at end of .0101635 second.
4th at end of .0148364 second. **Answer.**

Problem 8a: Find the frequency, in cycles per second, of an alternating-current, having the form of a sine-wave, that passes through its maximum value every $\frac{1}{120}$ of a second.

The periodic time of one cycle being $\frac{1}{60}=.0166666$, prove the values as given in the above answer are correct.

The bases of the alternating-current or alternating-pressure curves might be expressed directly in terms of the time in seconds or in fractions of a second; denoted by t, instead of by π.

If the frequency of the alternating quantity is 60 ∼ which means $.01666\frac{2}{3}$ second for the base of a complete cycle, the base of *one lobe* of an alternating-curve would be $\frac{1}{2}$ of $.01666\frac{2}{3}$* = $.008333\frac{1}{3}$ second.

Time in seconds is the independent variable and the values of the alternating-quantities, (pressure, current, or magnetic field) will be the dependent variable.

Suppose it is desired to find the average value of a sine-wave pressure, having a frequency of 60 cycles per second, between the limits of 0 and $\frac{1}{120}$ of a second; meaning the average of one lobe. $\frac{1}{120}=\frac{1}{2f}$

The pressure may be expressed by:

$e=\mathrm{E}_{\max}\sin(2\pi ft)$, and multiplying both members of the equation by dt, gives: $edt=\mathrm{E}_{\max}\sin(2\pi ft)dt$.

The area included between this curve and the X axis, or between the curve and the horizontal datum line is expressed by:

$$\int_0^{\frac{1}{120}} edt=\int_0^{\frac{1}{2f}}\mathrm{E}_{\max}\sin(2\pi ft)dt$$

$$=\mathrm{E}_{\max}\left[-\frac{\cos(2\pi ft)}{2\pi f}\right]_0^{\frac{1}{2f}}$$

$$=\frac{\mathrm{E}_{\max}}{2\pi f}[-(-1)-(-1)]$$

$$=\frac{2}{2\pi f}\mathrm{E}_{\max};$$

which divided by the base $\frac{1}{2f}$ will give the average pressure.

When $t=\frac{1}{120}$, $2\pi ft=\frac{120}{120}\pi=\pi=180^\circ$ and $\cos 180^\circ=-1$.
When $t=0$, $\cos(2\pi ft)=+1$

**See problem 5, page 58.*

EXAMPLE 9: If the maximum value of an alternating-current having a sine-wave form is 100 amperes, find its effective (or the square root of the mean of the squares) **value.**

DATA.

$I_{max} = 100$ amperes.

—O—

The general equation of a sine-curve is:

$y = \mathbf{A} \sin x$; (in which **A** denotes a constant)

x denotes a varying angle expressed in degrees, in radians, or as a function of time; y denotes the *instantaneous* values of the sine of the varying angle multiplied by the constant **A**. If $\mathbf{A = 1}$ a *true* sine-curve is expressed.* The general equation will be rewritten to apply to the given current as: $i = \mathbf{100} \sin x$, and is represented by the curve **OAB**, figure 20. If a new curve as **OCB** is drawn with ordinates whose values are obtained by squaring the corresponding instantaneous values of the sine curve, this represents a "*curve of squares*", of which the equation will be:

$$i^2 = \mathbf{100}^2 \sin^2 x$$

The area of this curve, between the limits of 0 and π will be expressed by $\int_0^\pi i^2 dx = \mathbf{100}^2 \int_0^\pi \sin^2 x\, dx$. The *elementary* area is $y^2\, dx$.

To eliminate complexity in the process of integration by the presence of a *power* of sin x, it is advisable to introduce some equivalent expression for $\sin^2 x$, such as $\sin^2 x = \frac{1}{2} - \frac{\cos 2x.}{2}$ See page 21, equation 22.

Then:

$$\text{Area of curve of squares} = 100^2 \int_0^\pi \left(\frac{1}{2} - \frac{\cos 2x}{2}\right) dx$$

$$= 100^2 \int_0^\pi \frac{dx}{2} - \frac{\cos 2x\, dx}{2}$$

$$= \frac{100^2}{2} \left[\Bigg[_0^{\pi} x - \Bigg[_0^{\pi} \sin 2x \right.$$

$$= \frac{100^2}{2} \left[(\pi - 0) - (0 - 0)\right]$$

$$= \frac{100^2}{2}\, \pi$$

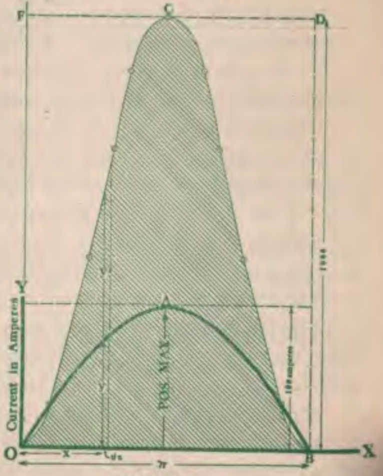

FIG. 20.

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*See page 50, and figure 16.

Dividing the above expression, by the base π, gives for the AVERAGE VALUE of the curve of squares; $\frac{\overline{100}^2}{2}=5000$.

The **R. M. S.**, of the square root of this value $=\sqrt{5000}=$ **70.7 amperes.**

The area of the curve of squares of a sine curve, is thus found to be:

$\frac{\pi}{2} I^2_{max}$; which is true for the curve of squares of any sine-curve.

The effective value of an alternating-pressure or current, is the square root of the average of the squares of the instantaneous values of the pressure or current; which also applies to an alternating pressure or current, of *any* shape.

If the alternating-pressure or current is a sine-wave in shape then the fixed relation between its "square root of the mean of the squares" and its "maximum" value is:

$$E_{eff} = \frac{E_{max}}{\sqrt{2}} = \frac{E^2_{max}}{2} = 0.707\ E_{max}$$

$$\text{or } I_{eff} = \frac{I^2_{max}}{2} = \frac{I_{max}}{\sqrt{2}} = 0.707\ I_{max}$$

This definite relation is true only when the pressures or currents are sine-waves.

The area of the *rectangle that includes one lobe of the curve of squares* of the sine-curve, may be expressed in terms of the maximum ordinate (I^2_{max}) of the curve of squares and the base π of the same curve, by:

πI^2_{max} (I_{max} refers to the sine-curve.)

The ratio of the two areas may be expressed as:

$$\frac{\text{area of curve of squares}}{\text{area of including rectangle}} = \frac{\frac{\pi}{2} I^2_{max}}{\pi I^2_{max}} = \frac{\pi}{2\pi} = \frac{1}{2}$$ Which is true for any sine-curve value.

Or the area of the curve of squares $=\frac{1}{2}$ *the area of the including rectangle. Consult figure* 20, *page* 63. In this figure the height of the rectangle should be = 10,000.

PROBLEM 9: Find the effective value of the current, in amperes, in an alternating-current circuit, in which the instantaneous value of the current is 141.4 amperes at 45° after the beginning of a positive lobe, assuming the shape of the current wave to be a sine-curve. **141.4 amperes. Answer.**

Problem 9a: Find the times, during one complete cycle, in fractions of a second, after the beginning of a positive lobe, at which a 60 cycle sine-wave alternating-pressure assumes its effective value.

PROBLEM 9b: Find the frequency in cycles per second, of a sine-wave alternating E. M. F. that assumes its effective value every $\frac{1}{200}$ *of a second.* $f=$ **50 cycles per second. Answer.**

Problem 9c: The base of one lobe of a sine-curve being 3.14159 inches long, and its maximum ordinate being 155.54 inches, compute the area, in square inches of the curve of squares, and of the rectangle including the curve of squares. Consult Plate I, page 1.

EXAMPLE 10: If the effective, or the Root of the Mean of the Squares, or R. M. S. value, of a sine-wave pressure is 100 volts, what is its maximum value? Find its amplitude factor.

The expression for the given conditions is:

$$E_{max} = \sqrt{2}\, E_{eff}$$

hence $E_{max} = 1.4142 \times 100 =$

$= \mathbf{141.42}$ **volts. Answer.**

DATA.

$E_{eff} = 100$ volts.

$\sqrt{2} = 1.4142$

———o———

To find its amplitude factor:

The amplitude factor of the sine-curve is defined as the ratio of the maximum value of the sine-curve, to its average value; or;

For any alternating-quantity the

$$\text{Amplitude factor} = \frac{\text{maximum}}{\text{average}} = \frac{E_{max}}{E_{av}} = \frac{141.42}{0.6366 \times 141.42} = 1.57. \ \textbf{Ans.}$$

The average value of a sine-curve is given in example 8, page 61, in terms of the *maximum* value, as $\frac{2}{\pi} I_{max}$.

$$\text{The } \textit{amplitude factor} \text{ must be } \frac{I_{max}}{\frac{2}{\pi} I_{max}} = \frac{\pi}{2} = \frac{3.14159}{2} = 1.570.$$

PROBLEM 10: Find the numerical value of the form factor of a sine-curve.

Form factor of a sine-curve is defined as the ratio of its effective value to its average value. **1.11. Answer.**

Problem 10a: If the base of one lobe of a sine-curve is 6.28 inches, and its maximum ordinate is 10 inches what is its area in square inches? Find the area, in square inches, of its curve of squares.

EXAMPLE 11: Given the equation of an alternating-pressure to be $e=141.4 \sin (377\, t)$; find its frequency; its effective value; its instantaneous value when it is changing at the rate of 2000 volts per second, the greatest rate, in volts per second, at which it changes; the value of the instantaneous pressure, in volts, at this instant, and the value of t, in seconds, when e first arrives at a negative maximum value, after the beginning of a positive lobe.

$$2\pi f = 377$$

$$f=\frac{377}{2\pi}=\mathbf{60\ cycles.}\quad \text{(very nearly)}$$

Answer.

DATA.

$E_{max}=141.4$ volts.

$E_{eff}=0.707\times 141.4$

$=999.69$ volts.

$E_{eff}=0.707\times 141.4=999.69$ volts. **Answer.**

By Calculus, the rate of change is expressed by:

$$\frac{de}{dt}=377\times 141.4 \cos (377\, t)$$

If the rate of change is 2000 volts per second then:

$$2000=377\times 141.4 \cos (377\, t)$$

From which; $\cos (377\, t)=\dfrac{2000}{141.4\times 377}=\dfrac{2000}{53307.8}=0.03751$

From tables of natural trigonometrical functions, the angle corresponding to the cosine 0.03751 is found to be 87° 51′, and the sine of 87° 51′ is found to be 0.99930.

Substituting this value of the sine in the original equation gives:

$e=141.4\times 0.99930=141.3$ volts; being the instantaneous value of the pressure, when the pressure is changing at the rate of 2000 volts per second.

To find the *maximum* rate of change, place the value of the first differential coefficient $\frac{de}{dt}=141.4\times 377 \cos (377\, t)=0$; then $\cos (377\, t) = 0$. When the cosine of $(377\, t)=0$ the sine of $(377\, t)=1$; the two curves being 90° or $\frac{\pi}{2}$ apart.

The pressure is *changing at a maximum rate* when the curve is passing through the horizontal datum line. The sine-curve of pressure crosses the horizontal datum line when $(377\, t)=180°$ or some multiple of 180°. The maximum rate of change occurs when the curve is passing from positive values to negative values or vice versa; when passing through zero or horizontal datum.

When passing through the horizontal datum, cos (377 t) = 1 or −1 depending on whether (377 t) is 0°, 180°, 360°, etc.

When the cosine is 1, $\frac{de}{dt} = 141.4 \times 377 = 53307.8$ or the pressure is changing at the rate of **53307.8 volts per second. Answer.** The value of the **instantaneous pressure at this instant is 0. Answer.** If the cosine is −1 then $\frac{de}{dt} = -53307.8$ or the change is opposite from that when cosine is +1. If in one case the direction is upward, in the other case it is downward.

The first negative maximum, after the beginning of a positive lobe, occurs at a time t', in seconds, that may be found as follows:

The periodic time is:

$$\mathbf{T} = \frac{1}{f} = \frac{1}{60} = \text{one-sixtieth of a second.}$$

The interval of time from the beginning of a cycle to the first **negative** maximum must be:

$$\frac{3}{4} \text{ of } \frac{1}{60} = \frac{3}{240} \text{ or } \frac{1}{80} \text{ of a second.}$$

Another method for finding the required interval would be to substitute for e in the original equation, the numerical value of the negative maximum, and solve the equation for t; as follows:

$$-141.4 = 141.4 \sin (377\ t)$$

From which; $\sin (377\ t) = -\mathbf{1}$

If the sine is $-\mathbf{1}$ it means that $(377\ t) = 270° = \frac{3}{2}\pi$ (in radians). But $377 = 2\pi f$.

Then:

$$2\pi f t = \frac{3}{2}\pi, \text{ and } t = \frac{\frac{3}{2}\pi}{2\pi \times 60} = \frac{3}{240} = \frac{1}{80} \textbf{ of a second.}$$

RATE OF CHANGE OF SINE-CURVE ALTERNATING-QUANTITIES.

The definition of alternating-quantities infers *"rate of change"*. In order that an E. M. F. be induced in a circuit, the magnetic field surrounding the circuit must undergo a *change*. The change in alternating-quantities is not uniform but variable and it is often important to know the rate of change at any given instant of time.

Suppose it is desired to ascertain the rate of change of a sine-current.

The expression for the current is:

$$i = \mathbf{I}_{\max} \sin(2\pi f t)$$

Differentiating gives:

$$d\,i = 2\pi f\,\mathbf{I}_{\max} \cos(2\pi f t)\,dt \qquad \text{[See page 33.]}$$

$$\text{But } \cos x = \sin(x + 90^\circ)$$

$$= \sin\left(x + \tfrac{\pi}{2}\right)$$

$$\text{and } \cos(2\pi f t) = \sin\left(2\pi f t + \tfrac{\pi}{2}\right)$$

Then:

$\frac{di}{dt} = 2\pi f \mathrm{I}_{\max} \sin\left(2\pi f t + \frac{\pi}{2}\right)$; in which $\frac{di}{dt}$ denotes the rate of change of the sine curve $i = \mathbf{I}_{\max} \sin(2\pi f t)$. The *"rate of change"* curve of a sine-curve may thus be said to be another sine-curve that is 90° or $\frac{\pi}{2}$ in advance of the given sine-curve.

PROBLEM 11: If the equation of an alternating-current is 282.8 sin (157.08 t), find its frequency; its effective value; its value when it is changing at the rate of 5000 amperes per second; its maximum value; its average value; the value of the time t in seconds, when i first reaches a negative maximum value after the beginning of a positive lobe, and the time interval, in seconds between two such values.

Answer. $f = 25 \sim$

Effective value 200 volts;

Instantaneous value when changing at the rate of 5000 amperes per second, 281.2 amperes;

Maximum value 282.8 volts;

Average value 179.8 amperes;

First negative maximum after $\frac{3}{100}$ second;

Interval between maxima values $\frac{1}{50}$ second.

Problem 11a: If the equation of a sine-wave of alternating-magnetic flux is $\phi = 14{,}140{,}000 \sin (377t)$, find the greatest rate, in maxwells per second, at which the flux is changing; the instantaneous value of the flux at this instant, and the instantaneous value when the flux is changing at the rate of 1000 megamaxwells per second. (In the equation as stated above, 14,140,000 means 14.14 megamaxwells; mega meaning one million).

EXAMPLE 12: Find the average and effective (square root of the mean of the squares) **values, the form-factor and the amplitude factor of a semi-circular wave shape, of alternating pressure, of current, or of magnetic field, given the maximum value of any of these to be $\frac{\pi}{2}$.**

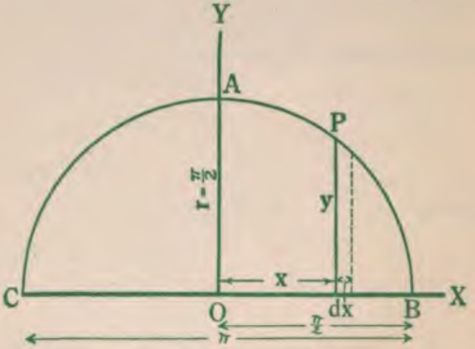

FIG. 21.

Let figure 21 denote the positive lobe of a semi-circular wave, whose maximum ordinate, $\mathbf{OA}=\mathbf{r}=\frac{\pi}{2}$, and whose base $\mathbf{CB}=\pi$; or $\frac{1}{2}$ the base $=\mathbf{OB}=\frac{\pi}{2}=\mathbf{r}$.

Assume the circle to be located with its center at the origin or intersection **O**, of the rectangular co-ordinate axes, **OX**, **OY**.

The equation of the circle when so located is:

$y=\pm\sqrt{r^2-x^2}$; in which x may denote angle in radians or time in fractions of a second, while y denotes values of pressure, current, or magnetic field.

BY CALCULUS.

As indicated in figure 21, an elementary area is ydx. Multiplying both sides of the above equation by dx gives:

$$ydx=\sqrt{\mathbf{r}^2-x^2}\,dx$$

and the area of the curve between the limits of 0 and $\frac{\pi}{2}$, or the **area of a quadrant,** is expressed by:

$$\int_0^{\frac{\pi}{2}} y\,dx = \int_0^{\frac{\pi}{2}} \sqrt{\mathbf{r}^2 - x^2}\,dx$$

$$= \Big[_0^{\frac{\pi}{2}} \tfrac{1}{2}(x\sqrt{\mathbf{r}^2 - x^2} + \mathbf{r}^2 \sin^{-1}\tfrac{x}{r}$$

(after substituting values of $\frac{\pi}{2}$ for both x and $\mathbf{r}$.)

$$= \tfrac{1}{2}(\tfrac{\pi}{2}\sqrt{\tfrac{\pi^2}{4} - \tfrac{\pi^2}{4}} + \tfrac{\pi^2}{4} \times \text{ angle whose sine is 1.})$$

which further reduces to:

$$\tfrac{1}{2}(0 + \tfrac{\pi^2}{4} \times 90^\circ)$$

but $90^\circ = \frac{\pi}{2}$; substituting this value, the area of a quadrant becomes:

$$= \tfrac{1}{2} \times \tfrac{\pi^2}{4} \times \tfrac{\pi}{2};$$

or $= \frac{1}{16}\pi^3$, **for the particular case in which** $\mathbf{r} = \frac{\pi}{2}$.

The area of the semicircle $= 2 \times \frac{\pi^3}{16} = \frac{\pi^3}{8}$

The area of *a quadrant* is therefore $\frac{\pi}{8}\mathbf{r}$ or $\frac{\pi}{4}\mathbf{r}^2$. (Since $\mathbf{r} = \frac{\pi}{2}$.)

The AVERAGE ORDINATE $= \frac{\pi^3}{\pi 8} = \frac{\pi^2}{8}$ (the area divided by the base)

$$= \frac{9.869}{8} = \mathbf{1.233}$$

To prove that the value obtained above is correct, solve the problem by *geometry*.

To find, by Calculus, the effective value:

The equation of the given curve being, $y = \sqrt{\mathbf{r}^2 + x^2}$ the equation of the curve of squares is

$$y^2 = \mathbf{r}^2 + x^2$$

The area of the curve of squares, between the limits of 0 and $\frac{\pi}{2}$ is expressed by:

$$\int_0^{\frac{\pi}{2}} y^2dx = \int_0^{\frac{\pi}{2}} (\mathbf{r}^2 - x^2)dx = \int_0^{\frac{\pi}{2}} \mathbf{r}^2dx - x^2dx.$$

$$= \left(\mathbf{r}^2x - \frac{x^3}{3} \right)_0^{\frac{\pi}{2}}$$

$$= \mathbf{r}^2\,\frac{\pi}{2} - \frac{\pi^3}{24} \text{ (and substituting for } \mathbf{r}\text{, its equal, } \tfrac{\pi}{2})$$

gives: $\int_0^{\frac{\pi}{2}} y^2dx = \frac{\pi^2}{4} \times \frac{\pi}{2} - \frac{\pi^3}{24};$

$$= \frac{\pi^3}{8} - \frac{\pi^3}{24} = \pi^3\left(\tfrac{1}{8} - \tfrac{1}{24}\right);$$

$= \frac{\pi^3}{8}$ = area of ½ lobe (corresponding to ½a semicircle).

Therefore the **area of one lobe of the curve of squares** is

$$= 2\frac{\pi^3}{12} = \frac{\pi^3}{6}$$

and the average value of the ordinates of the curve of squares is $= \frac{\pi^3}{\pi 6} = \frac{\pi^2}{6}$ (since $\mathbf{r} = \frac{\pi}{2}$, the average value of the curve of squares may be expressed in *terms of the radius* as $\frac{\pi}{3}\mathbf{r}$).

Can this be proved by methods of Geometry?

The **effective value** is $= \sqrt{\frac{\pi^2}{6}} = \frac{\pi}{\sqrt{6}} = \frac{3.1416}{2.449} = \mathbf{1.282.}$ **Answer.**

The *effective value* in terms of the *radius* is:

$= \sqrt{\frac{2}{3}}\,\mathbf{r}^2 = \frac{\sqrt{2}}{\sqrt{3}}\,\mathbf{r} = 0.8165\,\mathbf{r}$ for any circle whose radius is **r**.

The **form-factor** $= \dfrac{\text{effective value}}{\text{average value}} = \dfrac{\frac{\pi}{\sqrt{6}}}{\frac{\pi^2}{8}} = \dfrac{8}{\sqrt{6}\,\pi}.$

$$= \frac{8}{2.449 \times 3.1416} = \frac{8}{7.6327} = \mathbf{1.039.}\ \textbf{Answer.}$$

The **amplitude-factor** $= \dfrac{\text{maximum value}}{\text{average value}} = \dfrac{\frac{\pi}{2}}{1.233} = \dfrac{3.1416}{2.466} =$ **1.273. Answer.**

The amplitude-factor is sometimes called the "*peak-factor*".

It will be interesting to note in this connection the relation of the area of the "positive lobe" of this circular alternating quantity with the area of its including rectangle. The area of the including rectangle under the given conditions, is expressed by:

$$\pi \times \frac{\pi}{2} \text{ or } \frac{\pi^2}{2} = \frac{\pi}{\sqrt{2}} = 2.221$$

while the area of the semicircle is expressed by $\frac{\pi^3}{8}$.

The ratio of the two areas may be expressed as:

$$\frac{\text{area of semicircle}}{\text{area of including rectangle}} = \frac{\frac{\pi^3}{8}}{\frac{\pi^2}{2}} = \frac{\pi}{4} = 0.7854.$$

Or the area of any semicircle is $\frac{\pi}{4}$ times the area of the including rectangle.

The same relation exists between the area of a *circle* and the area of its *including square.*

The relation of the area of the curve of squares of the semicircle, to the area of the rectangle including the curve of squares, may be found in a similar manner, by comparing the area of the curve of squares expressed by $\frac{\pi^3}{6}$ with the area of the including rectangle expressed by

$$\left[\pi \times \left(\tfrac{\pi}{2}\right)^2\right] = \frac{\pi^3}{4}, \text{ as; } \frac{\frac{\pi^3}{6}}{\frac{\pi^3}{4}} = \frac{4}{6} = \frac{2}{3}.$$

The area of the curve of squares, for any semicircle, is only $\frac{2}{3}$ the area of its including rectangle.

PROBLEM 12: Find the effective value of a semicircular wave shape of alternating-pressure, whose maximum value is 120 volts.

97.9 volts. Answer.

Problem 12a: Find the instantaneous value of the current, expressed in amperes, in an alternating-current circuit in which the frequency is 60 cycles per second, and the wave shape is semicircular, at a time $\frac{1}{180}$ second after the beginning of a positive lobe of current.

HOW TO PLOT THE "CURVE OF SQUARES" FOR A CIRCLE.

If a circle has its center located at the intersection of the two axes of reference, its equation being $y=\sqrt{\mathbf{r}^2-x^2}$; the equation of its curve of squares is $y^2=\mathbf{r}^2-x^2$.

To make a definite example, the radius of the given circle will be assumed to be 4 inches.

The method of computing the ordinates of the curve of squares will be to assume certain values for x, and solve the equation of the curve of squares, for the corresponding values of y.

These are tabulated as follows:

When $x=0$, $y^2=16$
" $x=\frac{1}{5}$, $y^2=15.96$
" $x=\frac{2}{5}$, $y^2=15.80$
" $x=\frac{1}{2}$, $y^2=15.75$
" $x=\frac{4}{5}$, $y^2=15.36$
" $x=1$, $y^2=15$
" $x=1\frac{1}{5}$, $y^2=14.56$
" $x=1\frac{2}{5}$, $y^2=14.04$
" $x=\sqrt{2}$, $y^2=14$
" $x=\sqrt{3}$, $y^2=13$
" $x=1\frac{4}{5}$, $y^2=12.76$
" $x=2$, $y^2=12$
" $x=2\frac{1}{2}$, $y^2=9.75$
" $x=3$, $y^2=7$
" $x=3\frac{2}{5}$, $y^2=4.44$
" $x=3\frac{1}{2}$, $y^2=3.75$
" $x=3.87$, $y^2=1$
" $x=3\frac{4}{5}$, $y^2=0.76$
" $x=3.9$, $y^2=0.69$

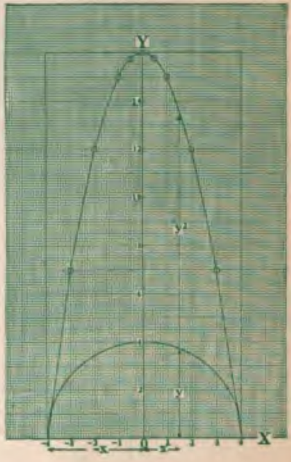

FIG. 22.

The curve of squares is plotted as shown in figure 22 by erecting, at distances from the origin 0 equal to the assumed values of x, perpendiculars equal to the corresponding computed values of y^2. y^2 being the product of $y\times y$ is in a true sense an area; but it is proper to represent an area by a line.

The values as tabulated apply to portion of the curve of squares to the *right* of the **Y** axis. The curve is symmetrical with reference to the

Y axis, so a corresponding portion could be plotted to the left of the Y axis. If *negative* values of x be substituted in the equation, the corresponding values of y^2 will be positive in sign.

A value -2 may be taken to illustrate; $-2^2 = +4$.

Then:

$y^2 = 16 - 4 = +12$; the same as obtained for $x = +2$.

The including rectangle is shown in dotted lines; the area of which is evidently 2 $(4 \times 16) = 128$ square inches. $128 = 8 \times 16 =$ base times the altitude.

The area included between the *curve of squares* and its base, the diameter of the circle, is obtained from:

$$2 \times \int_0^4 y^2 dx = 2 \times \int_0^4 r^2 dx - 2 \int_0^4 x^2 dx.$$

$$= 2r^2 \Big[_0^4 x - 2 \Big[_0^4 \frac{x^3}{3}.^*$$

$$= 2r^2 4 - 2 \times \tfrac{64}{3}$$

$$= 2 \times 64 = \tfrac{2}{3}\, 64$$

$$= 128 - \tfrac{128}{3}$$

$$= \tfrac{2}{3}\, 128$$

The factor 2 is introduced for the reason the area of ½ the curve of squares is found between the x limits of 0 and 4. Twice this area will give the whole area included by the curve of squares. The area of the including rectangle being 128 and the area included by the curve of squares being ⅔ of 128, the following relation must be true:

$$\frac{\text{Area of the curve of squares for a circle}}{\text{Area of the including rectangle}} = \frac{2}{3}.$$

This value will be found tabulated on page 92.

---

*The first term of this member of the equation expresses the area of the including rectangle. If $x = r$, the area of the including rectangle $= 2r^3$. From the area of the including rectangle is subtracted the area denoted by $\frac{2}{3}r^3$, (if $x = r$) to give the required area.

EXAMPLE 13: Find the form factor and the amplitude factor of a triangular wave shape.

$$\text{Form-factor} = \frac{\text{effective value}}{\text{average value}}.$$

$$\text{Amplitude-factor} = \frac{\text{maximum value}}{\text{average value}}.$$

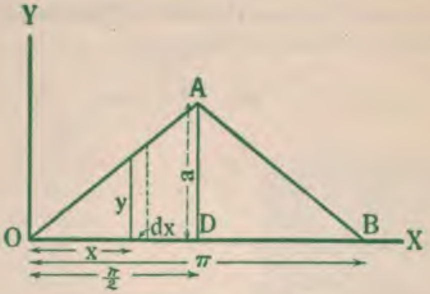

FIG. 23.

Let **OAB**, figure 23, denote a positive lobe of the triangular alternating wave, with **OB** = base = π; then **OD** = $\frac{\pi}{2}$.

By Calculus:

The equation of the line **OA** is

$$y = \mathrm{m}x$$

The elementary area $= y\,dx$. The triangle assumed being isosceles, the area of **OAD** = $\frac{1}{2}$ the area of the triangle **OAB**.

$$\text{The area of } \mathbf{OAD} = \int_0^{\frac{\pi}{2}} y\,dx = \int_0^{\frac{\pi}{2}} \mathrm{m}x\,dx$$

$$= \mathrm{m}\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} = \mathrm{m}\left(\frac{\frac{\pi^2}{4}}{4}\right. = \mathrm{m}\frac{\pi^2}{8}.$$

m denotes the tangent of the angle between the lines **OA** and the **X** axis (or the line OX); then

$\mathrm{m} = \frac{y}{x} = \frac{\mathbf{AD}}{\mathbf{OD}}$, calling the altitude of the triangle a, then $\mathrm{m} = \frac{\mathrm{a}}{\frac{\pi}{2}}$, (since **OD** = $\frac{\pi}{2}$).

Substituting this value for m, the area of $\frac{1}{2}$ the given triangle becomes

$$= \frac{a}{\frac{\pi}{2}} \times \frac{\pi^2}{8} = a\,\frac{\pi}{4}$$

The area of any triangle being $\frac{1}{2}$ the product of its base by its altitude, the last result may be readily checked.

It may be noted that the area of any triangle is equal to $\frac{1}{2}$ the area of its including rectangle.

And the **average value** $= \frac{a\pi}{4} \div \frac{\pi}{2} = \frac{a}{2}$ or $\frac{1}{2}$ the altitude.

To find the *effective value*, by the methods of Calculus:

The equation of the curve of squares is

$$y^2 = m^2 x^2$$

Multiplying both sides by dx gives $y^2dx = m^2x^2dx$.

The area of the curve of squares between the limits of **0** and $\frac{\pi}{2}$ **is expressed by:**

$$\int_0^{\frac{\pi}{2}} y^2dx = m^2 \int_0^{\frac{\pi}{2}} x^2dx$$

$$= m^2 \left[_0^{\frac{\pi}{2}} \frac{x^3}{3} = m^2 \left(\frac{\pi^3}{8 \times 3}\right)\right.$$

Substituting the value of $m = \frac{2a}{\pi}$; or $m^2 = \frac{4a^2}{\pi^2}$ gives the expression for the area;

$$= \frac{4a^2}{\pi^2} \times \frac{\pi^3}{24} = a^2\,\frac{\pi}{6}$$

The average value of the curve of squares is

$$= \frac{\frac{a^2\pi}{6}}{\frac{\pi}{2}} = \frac{2a^2}{6} = \frac{a^2}{3}.$$

The *square root* of the *average value* of the curve of squares or the

Effective value $= \sqrt{\frac{a^2}{3}} = \frac{a}{\sqrt{3}} = 0.577a.$ **Answer.**

Form factor $= \frac{\frac{a}{\sqrt{3}}}{\frac{a}{2}} = \frac{2}{\sqrt{3}} = \frac{2}{1.732} = 1.154.$ **Answer.**

Amplitude factor $= \frac{a}{\frac{a}{2}} = 2.$ **Answer.**

PROBLEM 13: Find the effective value, the form factor, and the amplitude factor of a triangular-wave having a base equal to π, *one side equal to 4 and the other side equal to 3.*

Consult page 30.
Angle opposite the side whose length is 3, is = 47°51′.
Altitude of triangle = 2.965. **Effective value = 1.71. Answer.**

Problem 13a: Find the effective value, the form factor, and the amplitude factor of a triangular-wave having a base of 4, one side of 5 and the other side of 3.

METHODS OF FINDING THE AREAS OF CURVES.

The average and effective values, as well as the form factor and amplitude factor or "peak" factor, all depend upon the areas of curves.

The author desires to call attention to five methods for finding the area of curves; meaning the area included between the curve, or certain portions of the curve, and the **X** axis, employing the system of rectangular co-ordinates.

First. If the *equation* of the curve is known, the area may be found by methods of Calculus.

Second. If the shape of the curve is that of some geometrical form, or can be divided into several portions, each portion having some regular or symmetrical geometrical form, the area may be found by ordinary rules applying to the particular geometrical forms.

Third. The area of figures having *irregular* shapes may be found by employing the "*planimeter*". This necessarily means that the curves or figures must be carefully plotted to scale on paper having a level surface. Good card-board or bristol-board is desirable for this purpose.

The average value of any curve may be found by dividing the value of the area obtained by the planimeter, by the base. The area and the base being expressed in corresponding units. To obtain the effective value, a new curve must be carefully constructed by taking for ordinates the *square* of each corresponding ordinate of the first curve, finding the area of the new curve, (called the curve of squares), with the planimeter, and dividing this value by the same base as in the first case. The SQUARE ROOT OF this "MEAN OF THE SQUARES" value, IS THE EFFECTIVE VALUE desired.

Fourth. The curve whose form-factor, amplitude factor, average or effective value is desired, is carefully plotted on section-paper that is divided into squares, both large and small, and the total number of squares, both large and small, included within the linear boundary of the curve, found by counting. A convenient kind of paper is that which is divided into larger squares 1 inch on a side, each square sub-divided into 100 smaller squares.

Fifth. The curve of any shape whatever may be carefully plotted on cardboard of uniform thickness, *within a* certain *rectangular area*. The rectangle and the curve may be carefully cut out with a small sharp thin knife-blade, and the parts carefully weighed on delicate chemical balances. The *areas* are to each other as the *weights*. The "curve of squares" may of course be constructed and treated by this method as well as the original curve.

EXAMPLE 14: Find the form factor and the amplitude factor of an elliptic-shape wave of alternating pressure or current.

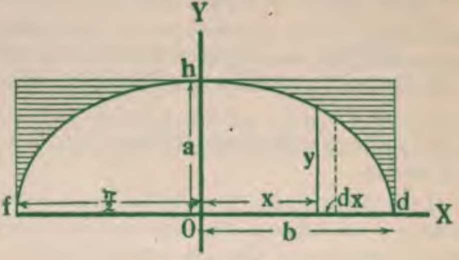

FIG. 24.

If the ellipse is referred to rectangular co-ordinate axes X and Y, as indicated in figure 24, the equation of the ellipse is:

$$y = \pm \tfrac{a}{b}\sqrt{b^2 - x^2}$$

in which the + sign denotes ordinates above the X axis, and the − sign denotes ordinates below the X axis. That is; for every value of x less than b, or $\frac{\pi}{2}$, y has *two* values; one +; the other of equal numerical value, but having the − sign. For the present discussion only the + values are considered. b denotes ½ the major axis; the whole axis is denoted by fd; and a denotes ½ the minor axis; denoted by Oh. (Oh is ½ the minor axis.) Also it should be noted that $b = \frac{\pi}{2}$ and the base of a complete lobe $= \pi$.

The above equation is obtained from: $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ as follows:

$$\frac{a^2x^2 + b^2y^2}{a^2b^2} = 1.$$

$$a^2x^2 + b^2y^2 = a^2b^2$$

$$b^2y^2 = a^2b^2 - a^2x^2$$

$$y^2 = \frac{a^2}{b^2}\,b^2 - \frac{a^2}{b^2}\,x^2$$

$$y^2 = \frac{a^2}{b^2}\,(b^2 - x^2)$$

$$\therefore y = \pm \frac{a}{b}\sqrt{b^2 - x^2}$$

The area of ½ the lobe, by Calculus, is expressed by

$$\int_0^{\frac{\pi}{2}} y dx = \frac{a}{b} \int_0^{\frac{\pi}{2}} \sqrt{b^2-x^2}\, dx;$$

$$= \frac{a}{b} \left[\frac{x}{2}\sqrt{b^2-x^2} + \frac{b^2}{2} \sin^{-1} \frac{x}{b} \right]_0^{\frac{\pi}{2}}$$

This may be expressed in terms of the versine if desired.

$$= \frac{a}{\frac{\pi}{2}} \left[\frac{\pi}{4}\sqrt{\frac{\pi^2}{4} - \frac{\pi^2}{4}} + \frac{\pi^2}{8} \times \text{the angle whose sine is} \frac{\frac{\pi}{2}}{\frac{\pi}{2}} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{2a}{\pi}\left[\frac{\pi^2}{8} \times \frac{\pi}{2}\right] = \frac{2a}{\pi} \times \frac{\pi^3}{16}$$

$$= \frac{a\pi^2}{8}$$

Then the average ordinate $= \frac{2a\pi^2}{8\pi} = \frac{\pi}{4} a = .7854a$

If $a = b$ then $y = \pm\sqrt{b^2 - x^2}$; or the ellipse becomes a circle whose area $= \pi b^2$; or πr^2.

The "*curve of squares*" for the ellipse is:

$$y^2 = \frac{a^2}{b^2}(b^2 - x^2)$$

It should be noted that for any value assigned to x, y has only *one*, and that a + value. All of the curve of squares is therefore *above* the X axis.

The *area* of this curve between the limits of 0 and $\frac{\pi}{2}$ is expressed by:

$$\int_0^{\frac{\pi}{2}} y^2 dx = \frac{a^2}{b^2} \int_0^{\frac{\pi}{2}} (b^2 - x^2) dx$$

$$= \frac{a^2}{b^2} \int_0^{\frac{\pi}{2}} b^2 dx - \int_0^{\frac{\pi}{2}} x^2 dx$$

$$= \frac{a^2}{b^2} \left[b^2 x \right]_0^{\frac{\pi}{2}} - \left[\frac{x^3}{3} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{b^2} \left(b^2 \frac{\pi}{2} - \frac{\pi^3}{24} \right)$$

Substituting for b^2, its value $\frac{\pi}{4}$ gives the area under discussion or $\frac{1}{2}$ the area of the curve of squares as:

$$= \frac{4a^2}{\pi^2} \left(\frac{\pi^2}{4} \times \frac{\pi}{2} - \frac{\pi^3}{24} \right) = \frac{4a^2}{\pi^2} \left(\frac{\pi^3}{8} - \frac{\pi^3}{24} \right)$$

$$= \frac{4a^2}{\pi^2} \times \frac{\pi^3}{12} = \frac{4b^2\pi}{12}$$

$$= \frac{\pi}{3} a^2$$

The area of the whole curve of squares, is $\frac{2}{3} \pi a^2$.

Dividing this area by the base π to find the average ordinate of the complete curve of squares gives:

Average ordinate of the curve of squares $= \frac{2}{3} a^2$.

Hence the square root of the mean of the squares or

The Effective value $= \sqrt{\frac{2}{3} a^2} = \frac{\sqrt{2}}{\sqrt{3}} a$. (Compare with semicircle on page 71.)

$$= \frac{1.4142}{1.7320} = \mathbf{0.816\,a}.$$

The form factor $= \frac{\text{effective value}}{\text{average value}} = \frac{\sqrt{\frac{2}{3}}a}{\frac{\pi}{4}a} = \frac{0.81651}{0.78539} = \mathbf{1.039}.$

The amplitude factor or the peak factor

$$= \frac{\text{maximum value}}{\text{average value}} = \frac{a}{\frac{\pi a}{4}} = \frac{4}{\pi}$$

$= \mathbf{1.273}$ (Compare with semicircle, page 71.)

It may be demonstrated that the foregoing values pertaining to an elliptical wave shape are true if the major axis is taken along the **Y** axis, instead of along the **X** axis, as in the present case.

PROBLEM 14: Find the effective value of the pressure, expressed in volts, of an elliptic shaped wave of alternating-pressure, whose maximum value is 120 volts. **97.9 volts. Answer.**

Problem 14a: Find the effective value of the current, expressed in amperes, of an elliptic shaped wave of alternating-current whose maximum value is 9.8 amperes.

Problem 14b: The average value of an elliptic shaped wave of alternating pressure is given as 78.54 volts; find its maximum and effective values.

Problem 14c: Find the relation of the area of an ellipse to its including rectangle. See figure 24, page 80, in which the shaded portion represents the *difference* between the area of the including rectangle and the area of the semi-ellipse.

EXAMPLE 15: Find the form factor and the peak factor of a parabolic wave of alternating-current, whose maximum value is 120 amperes.

If, as shown in figure 25, the curve is referred to the axis of ordinates, **OY** as an axis of symmetry; the vertex of the parabola being at **V** the equation of the curve will be expressed by:

(1) $x^2 = -py + m$ or solving for y;

(2) $y = \frac{m - x^2}{p}$

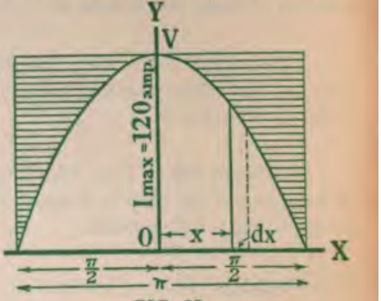

FIG. 25.

If y denotes *current* and x denotes *time*, the equation might be written:

(3) $i = \frac{m - t^2}{p}$

Since p and m are *constants*, it will be well to find their numerical values, when the maximum altitude of the curve is (as given) 120 amperes, and the base of one lobe of the curve is taken as π. 2π is taken as **the base** of all alternating time function quantities **for one complete cycle.**

m = square of $\frac{1}{2}$ the base, in any parabola located as in figure 25.

If the intersection of the rectangular co-ordinate axes is at **O**, it is known that when x (or t) = 0, y (or i) is equal to 120. Substituting these two values in (3) gives:

$$p = \frac{m}{120}$$

It is also evident that when y (or i) = 0, x (or t) = $\frac{\pi}{2}$

Substituting these values in (3) gives:

$$m = \frac{\pi^2}{4} = 2.4674$$

Substituting the value m = 2.4674 in $p = \frac{m}{120}$ gives:

$$p = \frac{2.4674}{120} = 0.020561$$

Substituting the values of m and p just obtained, in equation (3), the equation of this *particular current curve* is:

$$i = -48.6\,t^2 + 120$$

The area of *one lobe* of this curve is; (2 times the area between the limits of 0 and $\frac{\pi}{2}$ for one-half a lobe).

$$2\int_0^{\frac{\pi}{2}} i dt = 2\int_0^{\frac{\pi}{2}} -48.6\, t^2 dt + 2\int_0^{\frac{\pi}{2}} 120\, dt; \text{ (2 times } \tfrac{1}{2} \text{ a lobe)}$$

$$= -97.2 \left(\frac{\pi^3}{24} + 240\frac{\pi}{2} \right)_0^{\frac{\pi}{2}}$$

$$= -97.2 \times 1.2918 + 240 \times 1.57073;$$

$$= 251.3$$

As a check, one may find the area of this parabola from the fact that the area of any parabola is equal to $\frac{2}{3}$ of its including rectangle.

The including rectangle in this case is $\pi \times 120$ and $\frac{2}{3}\pi \times 120 = 251.32$.

If the result obtained by the method of Calculus is correct, then the relations of the area of *any* parabola to the area of its including rectangle may be found as follows:

THE AREA OF THE PARABOLA as given, is:

$$= 251.3$$

The area of its including rectangle is $120 \times \pi = 3.14159 \times 120 = 376.9$.

The ratio of these two areas may be expressed by

$$\frac{\text{area of parabola}}{\text{area of including rectangle}} = \frac{251.3}{376.9} = \frac{2}{3};$$

when $\pi = 3.14159$. Compare with result obtained for **Problem 14c.**

The AVERAGE VALUE of the current must be:

$$\frac{251.3}{\pi} = \frac{251.32}{3.14159} = \textbf{80 AMPERES}$$

TO FIND THE EFFECTIVE VALUE.

The equation of the curve of squares will be the square of the equation of the parabola, or

$i^2 = (-48.6t^2 + 120)^2$; which expanded gives:

$i^2 = 2361.94\, t^4 - 11664\, t^2 + 14400.$

See page 216 for values of π^5, π^4, π^2, etc.

The area of this curve, between the limits of 0 and $\frac{\pi}{2}$ (or $\frac{1}{2}$ a lobe) will be:

$$\int_0^{\frac{\pi}{2}} y^2 dx = 23619.6 \Bigg[_0^{\frac{\pi}{2}} \frac{t^5}{5} - 11664 \Bigg[_0^{\frac{\pi}{2}} \frac{t^3}{3} + 14400 \Bigg[_0^{\frac{\pi}{2}} t$$

$$= 2361.96 \left(\frac{306.019}{5\times 32}\right. - 11664 \left(\frac{31.006}{3\times 8}\right. + 14400 \left(\frac{3.1416}{2}\right.$$

$= \mathbf{12077.36}$*. The area of one lobe $= 2\times 12077.36 = 24154.72$

The *average* ordinate *of the curve of squares* will be the area 12077.36 divided by the base $\frac{\pi}{2}$ or:

$$\text{THE AVERAGE ORDINATE} = \frac{12077.36}{\frac{\pi}{2}} = 7688.6$$

The EFFECTIVE or R. M. S. VALUE will be $\sqrt{7688.6} = 87.6$ amperes.

THE FORM FACTOR will therefore be: $\dfrac{\text{effective value}}{\text{average value}}$

$= \frac{87.6}{80} = 1.095$. **Answer.**

THE PEAK FACTOR will be: $\dfrac{\text{maximum value}}{\text{average value}} = \dfrac{120}{80}$

$= 1.5$. **Answer.**

PROBLEM 15: Construct the positive lobe, accurately to scale, of a parabolic wave of pressure whose maximum value is 4 volts, and whose base is π seconds. Let 1 inch denote one volt, and 1 inch denote 1 second.

The equation:

$$e = \frac{m - t^2}{p}$$, expresses the relation in this case.

Assuming the axis of symmetry of the curve to coincide with the **Y** axis as in figure 25, page 84, the value of

$$p = \frac{m}{4}, \text{ and of } m = \frac{\pi^2}{4} = 2.4674$$

when the numerical value of $p = 0.6168$.

---

*The relation between the area of the curve of squares for a parabolic-wave and the area of its including rectangle is found from the special case here given.

The area of the including rectangle in this case is $120^2 \times \pi = 45238.896$.

The area of the curve of squares is $= 24154.72$.

Various instantaneous values may be computed from the equation:

$$e=\frac{2.4674-t^2}{0.6168}$$, by assuming any desired value for t, between 0 and $\frac{\pi}{2}$, and solving the equation for e.

The following have been calculated and tabulated:

When $t=0$, then $e=4$ volts.
" $t=\frac{\pi}{10}$, " $e=3.8$ "
" $t=\frac{\pi}{6}$, " $e=3.5$ "
" $t=\frac{\pi}{5}$, " $e=3.\dot{3}$ "
" $t=\frac{\pi}{4}$, " $e=3$ "
" $t=\frac{\pi}{3}$, " $e=2.1$ "
" $t=\frac{\pi}{2}$, " $e=0$ "

What is the frequency of this pressure?

Problem 15a: Find the instantaneous value of a parabolic wave of current, whose maximum value is 10 amperes, at a time 0.0033 seconds after the beginning of a positive lobe, the periodic time of one cycle being $\frac{1}{50}$ of a second.

The instantaneous value of *pressure*, of a parabolic wave of pressure, at a time $1\frac{2}{3}$ seconds after the beginning of a positive lobe, is $106\frac{2}{3}$ volts, if the *maximum value* of pressure is 120 volts.

If the area of a parabola is desired, in terms of its base and altitude, when located as indicated in figure 25, page 84, it may be found as follows:

Let **a** denote the altitude (maximum ordinate) and let **b** denote the base of the parabola. (**b** denotes the distance π in figure 25.)

From the general equation:

$$y=\frac{m-x^2}{p}$$

$$m=x^2+py$$

$$\text{and } p=\frac{m-x^2}{y}$$

When $x=0$, $y=a$, then $p=\frac{m}{a}=\frac{b^2}{4a}$,

$$\text{and}\quad m=\frac{b^2}{4}$$ (m = the square of $\frac{1}{2}$ the base)

The equation of the parabola, in terms of **a** and **b** then becomes:

$$y = \frac{\frac{b^2}{4} - x^2}{\frac{b^2}{4a}} = \frac{\frac{4ab^2}{4} - 4ax^2}{b^2}$$

$$= a - \frac{4a}{b^2}x^2 = a\left(1 - \frac{4}{b^2}x^2\right)$$

Multiplying both members of this equation by dx to form the elementary **area, gives**

$$\text{Elementary area} = y\,dx = a\,dx - \frac{4a}{b^2}x^2dx.$$

The area of ½ lobe of the curve, between the limits of **o** and $\frac{b}{2}$ will be **expressed by:**

$$\int_0^{\frac{b}{2}} y\,dx = \int_0^{\frac{b}{2}} a\,dx - \int_0^{\frac{b}{2}} \frac{4a}{b^2}x^2dx$$

$$= \Big[_0^{\frac{b}{2}} ax - \Big[_0^{\frac{b}{2}} \frac{4a}{b^2}\frac{x^3}{3}$$

(The inferior limit 0 causes the expression to become zero.)

$$= \frac{ab}{2} - \frac{4a}{b^2} \times \frac{b^3}{24}$$

$$= \frac{ab}{2} - \frac{4}{24}ab = \tfrac{1}{2}ab - \tfrac{1}{6}ab.^*$$

$$= ab(\tfrac{1}{2} - \tfrac{1}{6}) = ab\left(\frac{3-1}{6}\right)$$

$$= \tfrac{1}{3}ab$$

The area of one complete lobe of **the curve will be twice** the value **just obtained; or** ⅔ab.

This proves that the area of any parabola, located as in figure 25, page 84, **is ⅔ the** product of its base times its altitude; or ⅔ the area of its including rectangle.

The equation of the curve of squares of this parabola is:

$$y^2 = \frac{m^2 - 2mx^2 + x^4}{p^2}\text{, which multiplied by } dx\text{, becomes:}$$

$$y^2dx = \frac{m^2dx - 2mx^2dx + x^4dx}{p^2}$$

---

* × **b** denotes the including rectangle of the parabola.

Then the area of one half lobe; between the limits o and $\frac{b}{2}$, is expressed by:

$$\int_o^{\frac{b}{2}} y^2 dx = \frac{1}{p^2}\left(\int_o^{\frac{b}{2}} m^2 dx - \int_o^{\frac{b}{2}} 2mx^2 dx + \int_o^{\frac{b}{2}} x^4 dx\right).$$

$$= \frac{1}{p^2}\left[_o^{\frac{b}{2}} m^2 x - \left[_o^{\frac{b}{2}} 2m\frac{x^3}{3} + \left[_o^{\frac{b}{2}} \frac{x^5}{5}\right.\right.\right. .$$

$$= \frac{1}{p^2}\left(m^2\frac{b}{2} - 2m\frac{b^3}{3\times 8} + \frac{b^5}{5\times 32}\right.$$

Substituting for m its value $\frac{b^2}{4}$, for m^2 its value $\frac{b^4}{16}$, and for p^2 its value $\frac{b^4}{16a^2}$ the last equation becomes:

$$= \frac{16a^2}{b^4}\left(\frac{b^5}{32} - \frac{b^5}{48} + \frac{b^5}{160}\right)$$

$$= \frac{a^2}{b^4}\left(\frac{b^5}{2} - \frac{b^5}{3} + \frac{b^5}{10}\right)$$

$$= \frac{4}{15} a^2 b$$

The area of the complete lobe of the curve of squares is twice the value just obtained, or

$$\frac{8}{15} a^2 b$$

The average ordinate of this curve of squares is, therefore,

$$\frac{\frac{8}{15}a^2 b}{b} = \frac{8}{15}a^2$$

The effective or square root of this mean value is then:

$$= \sqrt{\frac{8}{15}a^2}$$

$$= 0.732a$$

This value is tabulated on page 92.

The effective value of any parabolic-wave is 0.732 times the height of the parabola its self.

NOTE.

The author desires to call attention to the fact that while curves such as the circle, the ellipse and the parabola are discussed and treated as though they are *alternating* time function curves insofar as their form factors, amplitude factors, and effective values are concerned, such curves are *not alternating functions,* as is the sine-curve or the cosine-curve, or curves formed by their combination.

This is evident by substituting for x in the equation of each of th "*conics*" mentioned, a value greater than π. Try this in the equation fo the ellipse on page 80, and in the equation of the circle on page 70.

If the equation of the curve is written in the form of a *time functio* using t in place of x, and the intercept on the X axis, or the base of the curv is assumed to be π, then substitute a value for t, *greater* than π, or a valu for t greater than $\frac{\pi}{2}$ if the Y axis is an axis of symmetry of the given curve

To illustrate this point, suppose a value *greater* than be substitute for t, in the equation for the parabola on page 84. If $t=\frac{\pi}{2}$, $t^2=2.4674$ then $i=0$. Any value greater than $\frac{\pi}{2}$ will give i a *negative* value, and th curve will never cross the X axis so long as the values for t are greater tha $\frac{\pi}{2}$. The curve may be a *time* function curve, but it is not an *alternatin* time function curve.

The circle and ellipse are not even time function curves. Discussin such curves can only be of value in familiarizing one with methods whic may be adopted in treating the more complex alternating functions.

A short resume of the foregoing discussion of areas of curves, may b effected by a comparison of the areas included between a semi-ellipse, parabola, a sine-curve, a triangle, and their *common* base; all of the curve having the same *altitude*.

For convenience suppose the common base is π, and the commo altitude is 100.

The area included by the elliptical-curve is:

$$\frac{\frac{\pi^2}{4}}{4}\times100=2.4674\times100=246.74 \qquad \mathbf{246.7}$$

The area included by the parabolic-curve is

$$\tfrac{2}{3}\pi\times100=209.43 \qquad \mathbf{209.4}$$

The area included by the sine-curve is

$$2\times100=200.00 \qquad \mathbf{200.0}$$

The area included by the triangular-wave is

$$\tfrac{1}{2}\pi\times100=157.07 \qquad \mathbf{157.0}$$

The foregoing has an important bearing on the construction o concrete or stone arch bridges. It is evident that for a given span an given height the *amount of material* in a bridge having an elliptic shape arch, is less than for any other shape.

EQUIVALENT SINE-CURVE.

The question of *equivalent* sine-curves will be mentioned later in connection with irregular alternating-curves, as an approximate expedient for determining the power factor, (or $\cos\phi$) of irregular shaped waves. It may be instructive to illustrate a method of finding a sine-curve that is equivalent to, either a rectangle, a semiellipse, a semicircle, or a parabola.

The term "*equivalent*" in such a connection needs some definite significance attached to it. Equivalent will be assumed to mean a sine-curve that has the same *effective*, (or the same "*square root of the mean of the squares*") value as the given curve has.

To be specific, suppose it is desired to find the necessary altitude or the maximum ordinate of a sine-curve so that the sine-curve shall have the same effective value as an elliptical-wave, that has an altitude of 100. The base of the two curves is to be the same and is assumed to be equal to π.

The effective value of the elliptical-wave is expressed by:

$0.816\times$maximum ordinate and is equal to 81.6.

The expression for the effective value of the sine-curve is:

$0.707\times$maximum ordinate. (See table, page 92.)

According to the assumed condition, the effective value of the elliptical-wave being equal to the effective value of the sine-wave:

$0.707\times$maximum ordinate$=81.6$;

∴ The maximum ordinate of the required sine-curve is:

$$\tfrac{81.6}{0.707}=1.414\times81.6=115.38.$$

The maximum ordinate of a sine-curve having the same effective value as a parabolic-wave, whose altitude is 100 and base π is:

$$73.2\times1.414=103.50$$

The maximum ordinate of a sine-curve having the same effective value as a rectangle whose altitude is 100 and whose base is π is, $100\times1.414=141.4$.

The maximum ordinate of a sine-curve having the same effective value as a semicircle, whose altitude is $\frac{\pi}{2}$ and whose base is π is:

$$0.816\times\frac{\pi}{2}=\frac{\text{maximum of the sine-curve}}{\sqrt{2}}$$

$=\sqrt{2}\times0.816\times\frac{\pi}{2}=1.153\times$radius of the circle$=1.78$ in the given case.

In the following table, the *base* of one lobe of the given curve is assumed to be π.

	Rectangle	Semiellipse	Semicircle	Sine-curve	Parabolic Curve	Triangle
Area of one lobe of the curve in terms of its maximum	$\pi \times (\text{max})$	$\frac{\pi^2}{4}(\text{max})^2$	$\frac{\pi(\text{max})^2}{2}$	$2 \times (\text{max})$	$\frac{2}{3}\pi(\text{max})$	$\frac{1}{2}\text{base} \times \text{max}$
Ratio of the area of the curve, to the area of its including rectangle	1	$\frac{\pi}{4}$ $= 0.7854$	$\frac{\pi}{4}$ $= 0.7854$	$\frac{2}{\pi} = 0.632$	$\frac{2}{3}$	$\frac{1}{2}$
Ratio of the average ordinate of the original curve to its maximum ordinate.	1	0.7854	0.7854	0.6366	0.666	0.500
Area of one lobe of the curve of squares in terms of the maximum ordinate of the original curve.	1	$\frac{2}{3}\pi(\text{max})^2$ base $= \pi$	$\frac{2}{3}\pi(\text{max})^2$ base $= \pi$	$\frac{\pi}{2} \times (\text{max})$ base $= \pi$	$\frac{8}{15}\pi(\text{max})^2$ base $= \pi$	$\frac{\pi}{6}(\text{max})^2$ base $= \pi$
Ratio of the area of the curve of squares to the area of its including rectangle.	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	0.533	$\frac{1}{6}$
Effective Value or the square root of the mean of the squares.	1	$0.816 \times \text{max}$	$0.816 \times \text{max}$	$\frac{1}{\sqrt{2}} \times \text{max}$	$0.732 \times \text{max}$	$0.577 \times \text{max}$
Form Factor or the ratio of the effective to the average value.	1	1.039	1.039	1.112	1.095	1.155
Ratio of the effective to the maximum.	1	0.816	0.816	0.707	0.732	0.577
Ratio of the average ordinate of the given curve to the effective	1	0.962	0.961	0.900	0.904	0.866
Average ordinate of the curve of squares in terms of the maximum of the given curve.	1	$\frac{2}{3}(\text{max})^2$	$\frac{2}{3}(\text{max})^2$	$\frac{1}{2}(\text{max})^2$	$\frac{8}{15}(\text{max})^2$	$\frac{1}{6}(\text{max})^2$

EFFECTIVE OR R. M. S. VALUES OF NON-SINE-PRESSURES.

EXAMPLE 16: Find the effective, or R. M. S. value of an alternating-pressure, which is composed of the sum of two simple sine-pressures whose two equations are as follows:

$$e_{\mathrm{I}} = \mathbf{141.4} \sin (377t)$$
$$e_{\mathrm{III}} = \mathbf{282.8} \sin (1131t)$$

DATA.

$2\pi f = 377^{*}$

$f = 60 \sim$

$\mathrm{E}_{\mathrm{I\,max}} = 141.4$ volts

$\mathrm{E}_{\mathrm{III\,max}} = 282.8$ volts

Adding the two pressures gives:

$$e_{\mathrm{I}} + e_{\mathrm{III}} = \mathbf{141.4} \sin (377t) + \mathbf{282.8} \sin (1131t).$$

Squaring this expression gives:

$$(e_{\mathrm{I}} + e_{\mathrm{III}})^2 = \mathbf{141.4}^2 \sin^2(377t) + 2 \times \mathbf{141.4} \sin (377t) \times \mathbf{282.8} \sin (1131t) + \mathbf{282.8}^2 \sin^2 (1131t).$$

The area of the curve of squares between the limits of 0 and $\frac{1}{2f}$ will be found by multiplying both members of the last equation by dt, and integrating the expression between the limits of 0 and $\frac{1}{2f}$.

The area is expressed by:

$$\int_0^{\frac{1}{2f}} (e_{\mathrm{I}}+e_{\mathrm{III}})^2 dt = \mathbf{141.4}^2 \int_0^{\frac{1}{2f}} \sin^2(377t)dt + \mathbf{282.8}^2 \int_0^{\frac{1}{2f}} \sin(377t) \times \sin(1131t)\, dt^{\dagger} + \mathbf{282.8}^2 \int_0^{\frac{1}{2f}} \sin^2(1131t)dt.$$

$$= \frac{1}{2f} \times \frac{\mathbf{141.4}^2}{\mathbf{2}}^{**} + \frac{1}{2f} \times \frac{\mathbf{282.8}^2}{\mathbf{2}}^{*} + 0. \qquad \text{(See page 63.)}$$

Dividing by the base $\frac{1}{2f}$ gives:

Average ordinate $= \frac{\mathbf{141.4}^2}{\mathbf{2}} + \frac{\mathbf{282.8}^2}{\mathbf{2}}$ = *mean of squares* of the resultant curve.

Therefore the **effective value**

$$= \sqrt{\frac{\mathbf{141.4}^2}{\mathbf{2}} + \frac{\mathbf{282.8}^2}{\mathbf{2}}} = \frac{\sqrt{\mathbf{141.4}^2 + \mathbf{282.8}^2}}{\sqrt{\mathbf{2}}}$$

$$= 0.707\sqrt{141.4^2 + 282.8^2}$$

$$= \mathbf{223.5 \text{ volts.}} \quad \textbf{Answer.}$$

---

*See table, page 209.

†The integral of this term is zero. (See equation on page 41.)

See **EXAMPLE 9, page 63.

The preceding example illustrates a very simple application of *Fourier's Theorem*. The application will be extended in the following problems. Time may be taken as the independent variable in these cases, and t is then the variable *factor* in the angle denoted by $(2\pi ft)$.

It may be observed that the *effective value* of a non-sine or irregular shaped alternating-pressure or current depends simply upon the magnitudes of the maxima values of its sine-components; or in other words, upon the amplitudes of its harmonies.

It may be stated as a general principle that the effective value of any alternating-current is the square root of the average of its squared instantaneous values. This is true whether the current curve is or is not a sine-curve in shape. Likewise the effective value of an alternating-pressure is the square root of its squared instantaneous values; being true for an alternating-pressure having an irregular shape as for a sine-curve pressure.

The *average value* of a non-sine alternating curve of pressure or current depends upon the relative *position* as well as upon the amplitudes of the sine-components. See figure 30, page 109.

PROBLEM 16: Find the effective value of an alternating-current, which is composed of the sum of three simple sine-currents, whose equations are:

$i_{\text{I}} = \mathbf{14.14} \sin(157\ t);$
$i_{\text{III}} = \mathbf{28.28} \sin(471\ t);$
$i_{\text{V}} = \mathbf{14.14} \sin(785t\ ).$ **24.46 amperes. Answer.**

Problem 16a: Find the effective value of an alternating-current, which is composed of the sum of three simple sine-currents, whose equations are:

$i_{\text{I}} = 10 \sin(\ 785\ t);$
$i_{\text{III}} = \ 2 \sin(2355\ t);$
$i_{\text{V}} = \ 3 \sin(3925\ t).$

Problem 16b: Plot the resultant curve mentioned in problem 16, adopting a scale of $7\frac{1}{2}$ inches $= \pi$ for the first harmonic component, and 1 inch $=$ 10 amperes.

The results found in example 16 and in problem 16 express an important principle or rule: **The effective value of a curve composed of the sum of a number of sine-curve is** obtained by finding **the square-root of the sum of one-half the squares of the maxima values of the component sine-curves;** or in other words, **the effective value of a curve, composed of the sum of a number of sine-curves, is equal to the square root of the sum of the squares of the maxima values of the component sine-curves, multiplied by 0.7071.** $(\mathbf{0.7071} = \frac{1}{\sqrt{2}})$

EXAMPLE 17: If an alternating - pressure is made up of the sum of the following sine-wave pressures:

$e_{I} = \mathbf{141.4} \sin(377t)$.

$e_{III} = \mathbf{282.8} \sin(1131t)$; **find its form factor and its CURVE-FACTOR.**

The expression for the given e. m. f. is:

$$e_{I} + e_{III} = \mathbf{141.4} \sin(377t) + \mathbf{282.8} \sin(1131t).$$

Its effective value as obtained in example 16, page 93, is **223.5** *volts*.

The **form factor is the** ratio; $\dfrac{\textbf{effective-value}}{\textbf{average value}}$.

The *average value* of the given e. m. f. is found as follows:

Multiplying both members of equation of the pressure by dt gives:

$$(e_{I} + e_{III})dt = 141.4 \sin(2\pi ft)dt + 282.8 \sin 3(2\pi ft)\,dt.$$

To find the area included by the curve for one-half cycle, integrate the equation between the limits of **0** and $\frac{\pi}{2}$; or since $\mathrm{T} = \frac{1}{f}$, between the limits expressed as **0** and $\frac{1}{2f}$

The area is expressed by;

$$\int_{0}^{\frac{1}{2t}} (e_{I} + e_{III})dt = 141.4 \int_{0}^{\frac{1}{2t}} \sin(2\pi ft)dt + 282.8 \int_{0}^{\frac{1}{2f}} \sin 3(2\pi ft)dt.$$

$$= 141.4 \left[_{0}^{\frac{1}{2t}} - \frac{\cos(2\pi ft)}{2\pi f} + 282.8 \left[_{0}^{\frac{1}{2t}} - \frac{\cos 3(2\pi ft)}{3(2\pi ft)}\right.\right.$$

Substituting the values of $\frac{1}{2f}$ and 0, for t gives:

$$\frac{141.4}{2\pi f} + \frac{282.8}{3(2\pi f)} = \text{superior limit, and}$$

$$-\frac{141.4}{2\pi f} - \frac{282.8}{3(2\pi f)} = \text{inferior limit.}\quad \text{Subtracting gives:}$$

$$\frac{141.4}{\pi f} + \frac{282.8}{3\pi f} = \text{numerical value of the required area.}$$

Dividing this expression by the assumed base $= \frac{1}{2f}$, gives:

$$\frac{\dfrac{3 \times 141.4 + 282.8}{3\pi f}}{\dfrac{1}{2f}} = \frac{2(3 \times 141.4 + 282.8)}{3\pi} = \frac{2 \times 707}{3\pi}$$

$= \mathbf{149}$; which is the average ordinate of the pressure curve.

The **Form Factor** is $= \frac{223.5}{149} = \mathbf{1.5}$. **Answer.**

The **CURVE FACTOR** is expressed by the ratio: $\frac{\text{effective - value of resultant curve}}{\text{maximum value of the } e_1 \text{ component}}$

$= \frac{223.5}{141.4} = \mathbf{1.58}$. **Answer.**

The curve factor is the ratio of the effective value of any curve **to the** ***maximum value*** of its *fundamental* or **first harmonic.**

Plate II, page 2, shows the relation of the component curves, and the resultant curve formed by adding the components.

PROBLEM 17: If an alternating-current is made up of the following component sine currents:

$i_{\mathrm{I}} = \mathbf{14.14} \sin(157t)$

and $i_{\mathrm{III}} = \mathbf{28.28} \sin(471t)$, *find its form factor, its amplitude factor, and its curve factor.*

Problem 17a: Find the form factor, the curve factor, and the amplitude factor of an alternating-current composed of the sum of the following sine-currents:

$$i_{\mathrm{I}} = \mathbf{14.14} \sin(157t);$$
$$i_{\mathrm{III}} = \mathbf{28.28} \sin(471t);$$
$$i_{\mathrm{V}} = \mathbf{14.14} \sin(785t).$$

The **amplitude factor is** the ratio $\frac{\textbf{maximum value}}{\textbf{average value.}}$

INSTANTANEOUS VALUES OF NON-SINE-PRESSURES AND CURRENTS.

EXAMPLE 18: Find the instantaneous value of the pressure, in volts, at a time 1.003 seconds after the beginning of a positive lobe, of an alternating pressure, composed of the SUM of the following sine-pressures:

$e_{\mathrm{I}} = \mathbf{141.4} \sin (377t);$
$e_{\mathrm{III}} = \mathbf{282.8} \sin (471t);$
$e_{\mathrm{V}} = \mathbf{141.4} \sin (785t).$

DATA.

$2\pi f_{\mathrm{I}} = 377$
$\therefore f_{\mathrm{I}} = 60$ cycles
$2\pi f_{\mathrm{III}} = 471$
$\therefore f_{\mathrm{III}} = 75$ cycles
$2\pi f_{\mathrm{V}} = 785$
$\therefore f_{\mathrm{V}} = 125$ cycles
$t = 1.003$ seconds

———O———

All three component sine-curves are assumed to start initially from the same point; (designated as **0**) all increasing positively. The phase angle between the three component curves is initially $= \mathbf{0}^{\circ}$.

If the phase angle between the zero values is 0° at the start, since the frequencies are not multiples of each other, the phase angle between other zero values is a variable.

The *sum* of the three curves may be expressed by:

$$e_{\mathrm{I}} + e_{\mathrm{III}} + e_{\mathrm{V}} = \mathbf{141.4} \sin (377t) + \mathbf{282.8} \sin (471t) + \mathbf{141.4} \sin (785t).$$

Substituting the value $t = 1.003$, and calling $f_{\mathrm{I}} = 60$, $f_{\mathrm{III}} = 75$, and $f_{\mathrm{V}} = 125$ cycles, the equation may be expressed as:

$$\begin{aligned} e_{\mathrm{I}} + e_{\mathrm{III}} + e_{\mathrm{V}} &= 141.4 \sin (2\pi \times 60 \times 1.003) + 282.8 \sin (2\pi \times 75 \times 1.003) \\ &\quad + 141.4 \sin (2\pi \times 125 \times 1.003) \\ &= 141.4 \sin (120.36\pi) + 282.8 \sin (150.45\pi) + 141.4 \sin \\ &\quad (250.75\pi). \\ &= 141.4 \sin 64^{\circ}\ 48' + 282.8 \sin 81^{\circ} + 141.4 \sin 135^{\circ}. \\ &= 141.4 \times .90483 + 282.8 \times .98769 + 141.4\ (+0.70711) \\ &= 127.94 + 279.31 + 99.98 \\ &= 507.24 \textbf{ VOLTS. Answer.} \end{aligned}$$

PROBLEM 18: Find the instantaneous value of the current, in amperes, at a time 1.003 seconds after the beginning of a positive lobe, of an alternating-current composed of the sum of the following sine-currents:

$i_{\mathrm{I}} = \mathbf{14.14} \sin 377t;$*
$i_{\mathrm{III}} = \mathbf{28.28} \sin 785t.$

---

*The values of frequencies corresponding to the numerical values of $2\pi f$ here given, may be found in table **I**, page 208.

The initial zero of both components and the resultant current all occur at the same instant. **32.79 amperes. Answer.**

Problem 18a: Find the instantaneous value of the pressure, in volts, at a time 1.003 seconds after the beginning of a positive lobe, of an alternating-pressure, composed of the sum of the following sine-pressures:

$$e_{I} = \mathbf{141.4} \sin 2\pi f t;$$
$$\text{and} \quad e_{III} = \mathbf{282.8} \sin 1131 t.$$

Let $f = 60$. Consult Plate II, page 2.

RESULTANT OF FOUR PRESSURES.

EXAMPLE 19: Find the resultant effective pressure in magnitude, and in phase relative to any one pressure, of four equal pressures, each 104 volts, effective value, with a phase angle of 30° between each.

The method employing the projections of the pressure vectors will be outlined.

DATA.

$E_1 = E_2 = E_3 = E_4 = 104$ volts.

$\phi = 30°$ between each pressures.

———O———

This method has the advantage of obtaining the value of the resultant, and the phase angle between the resultant and the datum or between the resultant and one of the components, in one operation.

The *algebraical sum* of the *horizontal* projections of any number of vectors (either current or pressure vectors) is always equal to the *horizontal* projection of their **resultant.**

The *algebraical sum* of the *vertical* projections of any number of vectors is always equal to the *vertical* projection of their **resultant.**

This may be understood better by referring to figure 27, page 100, in which the four pressures are denoted by $\mathbf{E_1}$, $\mathbf{E_2}$, $\mathbf{E_3}$ and $\mathbf{E_4}$; their resultant is designated by $\mathbf{E_R}$. According to the theory of the projections of these forces; the horizontal projections, $0a + 0b + \mathbf{E_1} = \mathbf{0c}$, and the vertical projections, $\mathbf{0d} + \mathbf{0e} + \mathbf{E_4} = \mathbf{f0}$.

f0 is the vertical projection of the resultant $\mathbf{E_R}$, and **0c** is the horizontal projection of the resultant $\mathbf{E_R}$. **f0** and **0c** are perpendicular to each other.

This method may be applied to Problem **19b**, page 101, as indicated in figure 28, page 102. In this case the horizontal projection of $\mathbf{E_4}$ being to the left of the vertical axis, must be *subtracted* from the horizontal projections to the right of the vertical axis; giving $\mathbf{0b} + \mathbf{0c} + \mathbf{E_1} - \mathbf{0a} = \mathbf{0d}$.

The horizontal projections will be designated as X projections.

The vertical projections will be designated as Y projections.

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Then according to figure 27 in which $\mathbf{E_1}$ is taken as the horizontal datum:

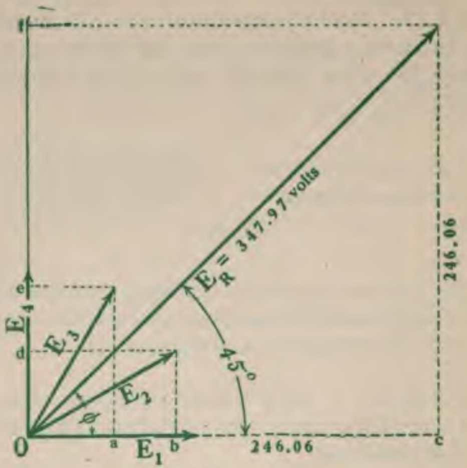


FIG. 27.

X PROJECTIONS.	Y PROJECTIONS.
$E_1 \cos 0^\circ = 104\times 1 = 104.00$	$E_1 \sin 0^\circ = 104\times 0 = 00.0$
$E_2 \cos 30^\circ = 104\times\frac{1}{2}\sqrt{3} = 90.06$	$E_2 \sin 30^\circ = 104\times\frac{1}{2} = 52.0$
$E_3 \cos 60^\circ = 104\times\frac{1}{2} = 52.$	$E_3 \sin 60^\circ = 104\times\frac{1}{2}\sqrt{3} = 90.06$
$E_4 \cos 90^\circ = 104\times 0 = 000.0$	$E_4 \sin 90^\circ = 104\times 1 = 104.$
Sum of the horizontal components $= 246.06$	Sum of the vertical components $= 246.06$

Then the resultant pressure, denoted by $\mathbf{E_R}$, will be expressed by:

$$E_R = \sqrt{\overline{246.06}^2 + \overline{246.06}^2} = \sqrt{2}\times 246.06 = 347.97 \text{ volts.} \quad \textbf{Answer.}$$

From figure 27, it is evident that $\tan\phi = 1$.

Therefore ϕ, the angle between the resultant pressure, ($\mathbf{E_R}$) and $\mathbf{E_1}$ is **45°**.

If counterclockwise direction is considered as positive, the resultant pressure $\mathbf{E_R}$ *leads* $\mathbf{E_1}$ by **45°**; it *leads* $\mathbf{E_2}$ by **15°**; it *lags* $\mathbf{E_3}$ by **15°**, and it *lags* $\mathbf{E_4}$ by **45°**.

The resultant pressure mentioned in Example 19, might be produced in practice by four alternators each producing equal sine-pressures, having their armatures so coupled together as to effect the required angular relations with each other, and having their armature terminals connected together *in series*, so that the pressures are cumulative; no one opposing any of the others.

While so-called effective values of alternating-pressures and alternating-currents are not strictly "*vector*" quantities, they are usually treated as such.

The relation may be best comprehended by considering a triangle of maxima values. If each side of such a triangle be divided by the $\sqrt{2}$ a *similar* triangle will result; but one having shorter sides; the triangle will now be a triangle of *effective* values of pressure.

A maximum value of alternating-pressure is a true vector quantity; having location, magnitude and direction. An effective (alternating) quantity may have location and magnitude, but is not considered as having *direction*.

PROBLEM 19: Find the resultant of four equal pressures each having an effective value of 110 volts, if the phase angle between each pressure is 45°. Find the phase angle of the resultant pressure with reference to each of the component pressures.

$E_R = 287.4$ volts; effective. *Leads* E_1 by 67.5°, *leads* E_2 by 22.5°, *lags* E_3 by 22.5°, *lags* E_4 by 67.5°. **Answer.**

Problem 19a: Find the resultant, effective value, of four pressures having the following effective values and phase relations. $E_1 = 110$ volts (datum); $E_2 = 55$ volts, leading E_1 by 30°; $E_3 = 110$ volts leading E_2 by 15°; $E_4 = 55$ volts leading E_3 by 30°.

Construct a vector diagram, to scale of 5 volts = 1 inch, showing the magnitudes and phase relations of *all* pressures in problem 19a.

Problem 19b: Suppose four alternators, each generating sine-wave pressures $E_1 = E_2 = E_3 = E_4 = 155.5$ volts maximum value, at 25 cycles per second, have their shafts coupled together as follows: 1 and 2 coupled so there is an angle of 30° between E_1 and E_2; 2 and 3 coupled for an angle of 45° between E_2 and E_3; 3 and 4 coupled for an angle of 60° between E_3 and E_4. Find the numerical value of the resultant pressure, maximum and effective, in volts, if the armature windings of the four machines are all connected together in series.

Figure 28 shows the relation of the four pressures, with the two axes X and Y on which the pressure vectors may be projected.

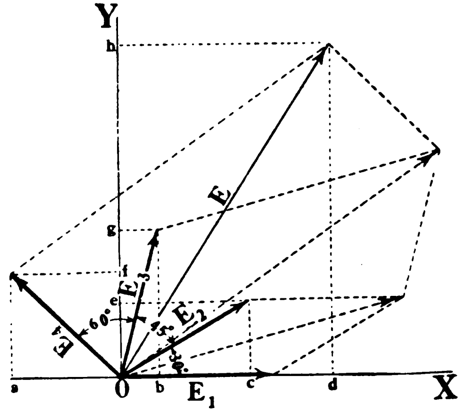


FIG. 28.

ADDITION OF SINE-PRESSURES.

EXAMPLE 20: Prove that the sum of two sine-curves, each having the same frequency but unequal amplitudes and with a certain phase angle between them, is also a sine-curve.

The method of proof to be adopted will be a mathematical one involving the trigonometrical functions, as follows:

The two component sine-curves will be denoted by two equations of the general form $y = \mathbf{A} \sin x$, and to be specific, two *sine-pressures* will be considered; both being referred to a sine-current, as datum.

The sine-current will be denoted by the equation:

$i = \mathbf{I} \sin (2\pi f t)$, and the two pressures by:
$e_1 = \mathbf{E}_1 \sin (2\pi f t + \phi_1)$ (1),
and $e_2 = \mathbf{E}_2 \sin (2\pi f t + \phi_2)$ (2).

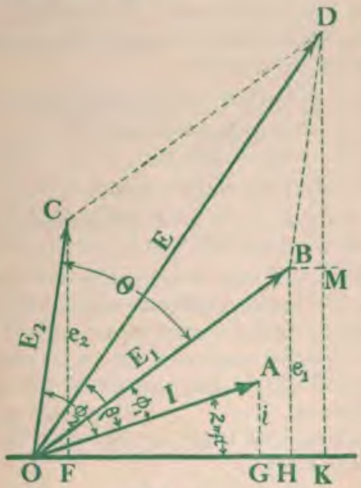


FIG. 29.

The relations of the various values expressed in the three equations are shown in figure 29, as vector quantities. **I**, $\mathbf{E}_1$ and $\mathbf{E}_2$ each denote the *maximum* value of each respective sine-curve. Plate III page 3 will illustrate the relations as *curves*.

I is denoted by the vector **OA**.
$\mathbf{E}_1$ is denoted by the vector **OB**.
$\mathbf{E}_2$ is denoted by the vector **OC**.

PLATE III shows the resultant of the sum of two sine-curves.

i, e_1 and e_2 each denotes the *instantaneous* value, at the same instant of time, of each respective sine-curve.

i is denoted by the vertical dotted line **GA**;
e_1 is denoted by the vertical dotted line **HB**;
e_2 is denoted by the vertical dotted line **FC**, in figure 29.

It is to be noted that both of the component pressures *lead* the current in phase, that they are not equal to each other, and that the angle between **I** and the horizontal datum is $(2\pi ft)$.

Expanding the two pressure equations (1) and (2) gives:

$$e_1 = \mathbf{E}_1 [\sin(2\pi ft)\cos\phi_1 + \cos(2\pi ft)\sin\phi_1] \text{ and}$$
$$e_2 = \mathbf{E}_2 [\sin(2\pi ft)\cos\phi_2 + \cos(2\pi ft)\sin\phi_2]. \quad \text{(Consult page 13)}$$

Adding these two equations gives:

$$e_1 + e_2 = \mathbf{E}_1 \sin(2\pi ft)\cos\phi_1 + \mathbf{E}_1\cos(2\pi ft)\sin\phi_1 + \mathbf{E}_2\sin(2\pi ft)\cos\phi_2 + \mathbf{E}_2\cos(2\pi ft)\sin\phi_2;$$ which by slightly rearranging becomes:

$$= \sin(2\pi ft)[\mathbf{E}_1\cos\phi + \mathbf{E}_2\cos\phi_2] + \cos(2\pi ft)[\mathbf{E}_1\sin\phi_1 + \mathbf{E}_2\sin\phi_2]. \quad (3)$$

$$\text{Let}\left\{\begin{array}{l}\mathbf{E}\cos\beta = \mathbf{E}_1\cos\phi_1 + \mathbf{E}_2\cos\phi_2;\text{ and}\\ \mathbf{E}\sin\beta = \mathbf{E}_1\sin\phi_1 + \mathbf{E}_2\sin\phi_2.\end{array}\right\} \quad (4).$$

Then by substituting the values assumed in (4), in equation (3); gives:

$$e_1 + e_2 = \mathbf{E}\sin(2\pi ft)\cos\beta + \mathbf{E}\cos(2\pi ft)\sin\beta.$$
$$= \mathbf{E}\sin(2\pi ft + \beta).^* \quad (5).$$

which is the equation of a sine-curve, having the same frequency as the component curves, but having a different maximum value, denoted by **E**, and a different phase angle, (referred to the current vector) denoted by β, (β is a constant) than the component curves, (1) and (2).

It should be noted that the last equation is of the general form:

$y = \mathbf{A}\sin x$. [**A** denotes **E** and x denotes $(2\pi ft+\beta)$]

The foregoing treatment might be applied to two current curves referred to one pressure curve.

PROBLEM 20: Find the numerical value of the maximum ordinate, **E**, *of the pressure curve resulting from adding the following sine-pressures; both being referred to a current curve, having the same frequency.*

$e_1 = 141.4 \sin(377t + 15°)$;
$e_{III} = 282.8 \sin(377t + 30°)$. **421 volts; 10° with $\mathbf{E}_2$. Answer.**

Also find the phase angle between the resultant and the e_1 component. (Consult figure **a**, **PLATE III**, page 3.)

---

*This may be seen from the trigonometrical relations of the portions of figure 29, where **DM** $= e_2$ and **MK** $= e_1$; then **E** $\sin(2\pi ft+\beta) =$ **DK** $= e_1 + e_2$.

Problem 20a: Find the maximum value of the pressure curve resulting from the addition of the following sine-wave pressures:

$$e_{\mathrm{I}} = 141.4 \sin (377t + 15^\circ);$$
$$e_{\mathrm{III}} = 282.8 \sin (377t + 30^\circ);$$
$$e_{\mathrm{V}} = 141.4 \sin (377t + 15^\circ).$$

Find the angles between the resultant and each component.

Problem 20b: Find the maximum value of the pressure curve resulting from the addition of two sine-pressure components, expressed by the equation:

$$e' = \mathbf{100} \sin (2\pi f t)$$
$$\text{and } e'' = \mathbf{200} \sin (2\pi f t + \mathbf{15}^\circ)$$

Consult figure a, **PLATE III**, page 3.

Problem 20c: Find the numerical value of the maximum of the pressure curve resulting from the addition of two sine-pressure components, expressed by:

$$e' = \mathbf{100} \sin (2\pi f t)$$
$$\text{and } e'' = \mathbf{100} \sin (2\pi f t + \mathbf{30}^\circ)$$

Consult figure b, **PLATE III**, page 3.

ADDITION OF SINE-PRESSURES.

EXAMPLE 21: Find an expression for the value of the resultant, when two sine-pressures, having the same frequency, but different numerical values, and different phase angles with reference to the current, are added together.

This is a continuation of the analysis of **EXAMPLE 20**, page 103.

By *multiplying* together the two equations numbered (4), on page 104, the following is obtained:

$$\mathbf{E}^2 \sin\beta\cos\beta = \mathbf{E}_1{}^2 \sin\phi_1 \cos\phi_1 + \mathbf{E}_2{}^2 \sin\phi_2 \cos\phi_2 + \mathbf{E}_1\,\mathbf{E}_2 \sin\phi_1 \cos\phi_2 + \mathbf{E}_1\mathbf{E}_2 \sin\phi_2 \cos\phi_1.$$

By *squaring each* equation given under (4), page 104, and then adding the results, the following is obtained:

$$\mathbf{E}^2 \cos^2\beta = \mathbf{E}_1{}^2 \cos^2\phi_1 + \mathbf{E}_2{}^2 \cos^2\phi_2 + 2\,\mathbf{E}_1\,\mathbf{E}_2 \cos\phi_1 \cos\phi_2;$$

and $\mathbf{E}^2 \sin^2\beta = \mathbf{E}_1{}^2 \sin^2\phi_1 + \mathbf{E}_2{}^2 \sin^2\phi_2 + 2\,\mathbf{E}_1\,\mathbf{E}_2 \sin\phi_1 \sin\phi_2;$

Adding the above two equations gives:

$$\mathbf{E}^2 (\sin^2\beta + \cos^2\beta) = \mathbf{E}_1{}^2 (\sin^2\phi_1 + \cos^2\phi_1) + \mathbf{E}_2{}^2 (\sin^2\phi_2 + \cos^2\phi_2) + 2\,\mathbf{E}_1\,\mathbf{E}_2 (\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2).$$

From which is obtained the following: (since $\sin^2\beta + \cos^2\beta = \mathbf{1}$).

$$\mathbf{E}^2 = \mathbf{E}_1{}^2 + \mathbf{E}_2{}^2 + 2\mathbf{E}_1\mathbf{E}_2 (\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2).$$

Which still further reduces to:

$\mathbf{E}^2 = \mathbf{E}_1{}^2 + \mathbf{E}_2{}^2 + 2\mathbf{E}_1\,\mathbf{E}_2 \cos(\phi_2 - \phi_1).$ (See equation (9) page 14).

Since $(\phi_2 - \phi_1) = \theta$, (See figure 29, page 103); the $\cos(\phi_2 - \phi_1) = \cos\theta$.

Taking the square root of both members of this equation gives:

$\mathbf{E} = \sqrt{\mathbf{E}_1{}^2 + \mathbf{E}_2{}^2 + 2\mathbf{E}_1\,\mathbf{E}_2 \cos\theta}$; (E, $\mathbf{E}_1$ and $\mathbf{E}_2$ may denote either *maxima* or *effective* values; the resultant being of corresponding designation).

Which is the same as the equation expressing the value of the resultant of two forces which act with an angle θ between them.

While the foregoing treatment deals with *maxima* values, it applies also to effective values; which may be shown as follows:

Applying the relation: $\mathrm{E}_{\mathrm{eff}} = \dfrac{\mathrm{E}_{\mathrm{max}}}{\sqrt{2}}$; or $\mathrm{E}^2_{\mathrm{max}} = 2\mathrm{E}^2_{\mathrm{eff}}$

$$\mathrm{E}_{\mathrm{eff}} = \frac{\mathrm{E}_{\mathrm{max}}}{\sqrt{2}} = \sqrt{2\,\mathrm{E}_1{}^2{}_{\mathrm{eff}} + 2\mathrm{E}_2{}^2{}_{\mathrm{eff}} + 4\mathrm{E}_{1\,\mathrm{eff}}\,\mathrm{E}_{2\,\mathrm{eff}}\cos\theta}$$

$$\text{or}\quad \sqrt{2}\,\mathrm{E}_{\mathrm{eff}} = \sqrt{2}\sqrt{\mathrm{E}_1{}^2{}_{\mathrm{eff}} + \mathrm{E}_2{}^2{}_{\mathrm{eff}} + 2\mathrm{E}_{1\,\mathrm{eff}}\,\mathrm{E}_{2\,\mathrm{eff}}\cos\theta}$$

Whence: resultant $\mathrm{E}_{\mathrm{eff}} = \sqrt{\mathrm{E}_1{}^2{}_{\mathrm{eff}} + \mathrm{E}_2{}^2{}_{\mathrm{eff}} + 2\mathrm{E}_{1\,\mathrm{eff}}\,\mathrm{E}_{2\,\mathrm{eff}}\cos\theta}$

In the case of the two pressures given, if the two component pressures are *equal* to each other, and the angle between them is 0°, then

$$\mathrm{E} = \sqrt{4\,\mathrm{E}_1{}^2} = 2\mathrm{E}_1 \quad (\cos 0° = 1)$$

If the two pressures are equal to each other and the angle between them is **90°** then:

$$\mathrm{E} = \sqrt{2}\;\mathrm{E}_2 = \sqrt{2}\;\mathrm{E}_1 \quad (\cos 90° = 0)$$

This is applicable when discussing "two-phase" alternating-currents.

If the two pressures are *equal to each other* and the angle between them is 180°: $\mathrm{E} = \sqrt{2\,\mathrm{E}_1{}^2 - 2\mathrm{E}_1{}^2} = 0$; $(\cos 180° = -1)$.

If the two pressures are equal to each other and the angle between them is 120° then:

$$\mathrm{E} = \sqrt{2\mathrm{E}_1{}^2 - 2\mathrm{E}_1{}^2(-\tfrac{1}{2})} = \sqrt{2\mathrm{E}_1{}^2 + \mathrm{E}_1{}^2} = \sqrt{3}\,\mathrm{E}_1$$

This will be applied when discussing "three-phase" alternating-currents.

It may be noted; that

$$\tan\theta = \frac{\mathrm{E}\ \sin\theta}{\mathrm{E}\ \cos\theta} = \frac{\mathrm{E}_1\ \sin\phi_1 + \mathrm{E}_2\ \sin\phi_2}{\mathrm{E}_1\ \cos\phi_1 + \mathrm{E}_2\ \cos\phi_2}$$

The pressures denoted by E, E_1 and E_2 are here considered as *maxima* values.

The foregoing discussion of the addition of sine-pressures, applies to *sine-currents*; the two component currents being denoted by:

$$i_1 = \mathrm{I}_1 \sin\,(2\pi f t + \phi_1);$$
$$i_2 = \mathrm{I}_2 \sin\,(2\pi f t + \phi_2).$$

The *resultant* current (either maximum or affective) is expressed by:

$\mathbf{I}=\sqrt{\mathbf{I}_1^2+\mathbf{I}_2^2+2\mathbf{I}_1\mathbf{I}_2\cos\theta}$; in which θ denotes the angle between $\mathbf{I}_1$ and $\mathbf{I}_2$.

In all of the preceding discussion of the addition of sine-curve quantities, $(2\pi ft)$ denotes a variable angle, while θ denotes a constant angle.

PROBLEM 21: Find the value of the angle, in degrees, between the effective values of two currents expressed by:

$$i_{\mathrm{I}}=14.14\sin(157t+15^\circ),\text{ and}$$
$$i_{\mathrm{III}}=28.28\sin(157t+30^\circ);$$

Also find the angle between their **resultant** *sum*, (effective) and the effective value of the i_{I} *curve*. If desired these curves may be referred to a pressure as datum. **15°; 10°** (approx.) **Answer.**

Problem 21a: Find the value of the angle, in degrees, between the maxima values of two sine-currents, expressed by:

$$i_{\mathrm{I}}=14.14\sin(94.24t+15^\circ),$$
$$\text{and}\quad i_{\mathrm{III}}=28.28\sin(94.24t+45^\circ).$$

The foregoing may be studied more at length by consulting plate IV, page 4, remembering that the results here shown are applicable alike to pressures and currents. The large curve in this figure is the resultant obtained by adding together three sine-curves which are not in phase with each other; but having a phase displacement of 15°. One of the sine-curves is taken as datum, and all other curves referred to it in phase relation. Had the three sine-curves been in phase, the maximum value of the resultant would have been 16 instead of about 15.6. The greater the phase displacement of the component sine-curves, the less the numerical value of the resultant maximum. This may be studied from the stand-point of vectors by consulting figure 29, page 103. The greater the angle between $\mathbf{E}_1$ and $\mathbf{E}_2$, the less the length of their resultant $\mathbf{E}$. The same principle applies to the resultant of *four* sine-wave pressures as shown in figure 28, page 102.

NON-SINE ALTERNATING CURVES.

While the curve resulting from the addition of two or more sine-curves having the *same frequency*, but different maxima values and phase position are sine-curves, curves resulting from the addition of several sine-curves of *different* frequencies are not sine-curves, but are irregular in shape, regardless of amplitude or phase positions. The positive lobes of such curves may or may not be similar to the negative lobes.

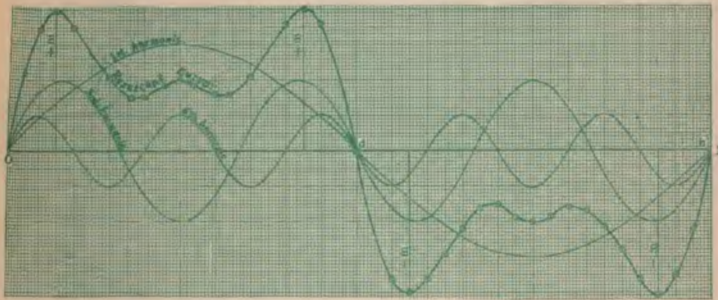


FIG. 30.

Figure 30, on this page, shows one form of alternating-pressure, (or current) resulting from adding three simple sine-curves together. An infinite number of different curves might result from adding together three sine-curves. The curve shown in figure 30, is the resultant of three sine-curves, having 1, 3 and 5 times the frequency of the resultant, and having maxima values of 3, 2 and 1 respectively, relative to the component sine-curve having 5 times the frequency of the resultant; designated in the figure as the **5th harmonic.**

If the three sine-curves had had *maxima values* in the *reverse* order given; namely, **1st,** 1; **2nd,** 2, and **3rd,** 3, and the three curves added, a very different resultant curve would have been found. It is thus seen that a large variety of shapes may be found for the resultant curve, by simply varying *magnitudes*. Greater variations result from varying *frequencies* and relative *position* of the component sine-curves.

Fourier, in investigating this fact discovered that a certain number of sine-curves of different frequencies, maxima values, and phase position may be found and added together to form the given irregular periodic, or *alternating-curve*, and formulated a systematic method for finding the numerical values of the frequencies and maxima values of the component sine-curves.

The method of finding the component sine-curves that unite to form any given irregular alternating-curve, is what may be called the *analysis* of the given curve. This process is exactly the reverse of the process of adding together a given number of definite sine-curves to form a resultant periodic curve. The latter process is "*synthesis*", and is a much simpler operation than that of analyzing a given curve to evaluate its proper sine-components.

Many important facts may however be ascertained by the graphical process of constructing alternating-curves from given sine-curves; either by *adding* sine-curves together or by *multiplying* them by each other.

While any periodic irregular shaped wave may be made up of the sum of sine-waves having frequencies that are both an even and an odd number of times the frequency of the fundamental or resultant curve, in ordinary alternating-current practice the irregular periodic pressure and current curves are composed only of the *odd* sine-components; or as sometimes stated only of the odd *harmonics*.

One reason why this condition obtains may be noted as follows:

When an alternating-pressure is induced by moving a wire (inductor) from one magnetic field to another, the direction of the pressure changes from a positive to a negative value when the wire passes from a field having a certain direction to a field having the opposite direction.

Under such conditions all the components (harmonics) of the pressure must undergo directional changes in unison with and similar to their resultant pressure.

By adding together two or more sine-waves having *odd* frequencies; one of the sine-curves having 3, 5 or 7 times the frequency of the other, it will be seen that when the resultant curve passes from a positive value to a negative one, (at the end of a positive lobe) all the components do likewise. See figure 30, page 109.

By adding together two or more sine-waves having *even* frequencies; 2, 4, 6, 8 etc., times one sine-curve, sometimes call the fundamental, it may be seen that when the resultant curve passes downward from positive to negative, the components pass upward from negative to positive. From a physical standpoint the latter phenomena does not coincide with the principle of induced pressures. A careful study of pages 47 to 53 will assist in an understanding of the principle. The elimination of the even harmonics in alternating-currents, renders the study of irregular periodic curves 50% less difficult; but it is no joke as it is.

PRODUCT OF TWO SINE-CURVES HAVING THE SAME FREQUENCY.

The theory involved in multiplying two sine-waves or two sine-curves together, forms the basis of calculating the power in alternating-current circuits.

There are two possible cases that should be considered:

I. *When the sine-curves are in phase with each other.*

II. *When the sine-curves are not in phase with each other.*

All sine-curves involved in this consideration are assumed to have the same frequency.

The resultant of the product of two sine-curves is in reality a curve formed by plotting the *products* of the corresponding *instantaneous values* of the two sine-curves.

Since it is not consistent to multiply two pressures together; or one current by another current, but more in the line of actual physical fact to multiply a pressure by a current, giving as a result a *power* value, the general discussion that immediately follows, will assume that a sine-pressure will be multiplied by a sine-current having the same frequency and the same phase. That is to say, the zero and the maxima values of both current and pressure occur at the same instant of time. The maxima values of the two curves do not necessarily have the same numerical value.

Let the equations of the two curves be:

$$e = \mathbf{E}_{\mathbf{max}} \sin (2\pi f t),$$
$$\text{and } i = \mathbf{I}_{\mathbf{max}} \sin (2\pi f t).$$

The product of the ordinates of these two curves at any instant, gives the *instantaneous power*, and is expressed by:

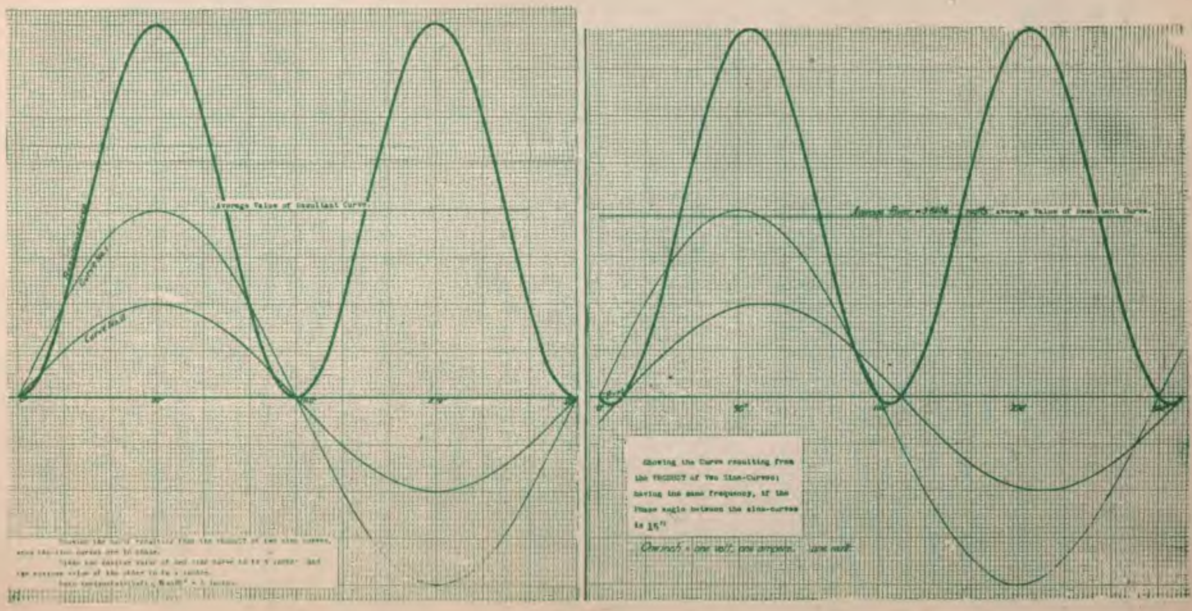
$$ei = \mathbf{E}_{\mathbf{max}} \mathbf{I}_{\mathbf{max}} \sin^2 (2\pi f t).$$

Denoting the first member by p, and expanding the second member, gives: (See page 21 for value of $\sin^2 x$)

$$p = \mathbf{E}_{\mathbf{max}} \mathbf{I}_{\mathbf{max}} \left[\frac{1}{2} - \frac{\cos 2\,(2\pi f t)}{2}\right].$$

$$= \frac{\mathbf{E}_{\mathbf{max}} \mathbf{I}_{\mathbf{max}}}{2} \left(1 - \cos 2\,(2\pi f t)\right).$$

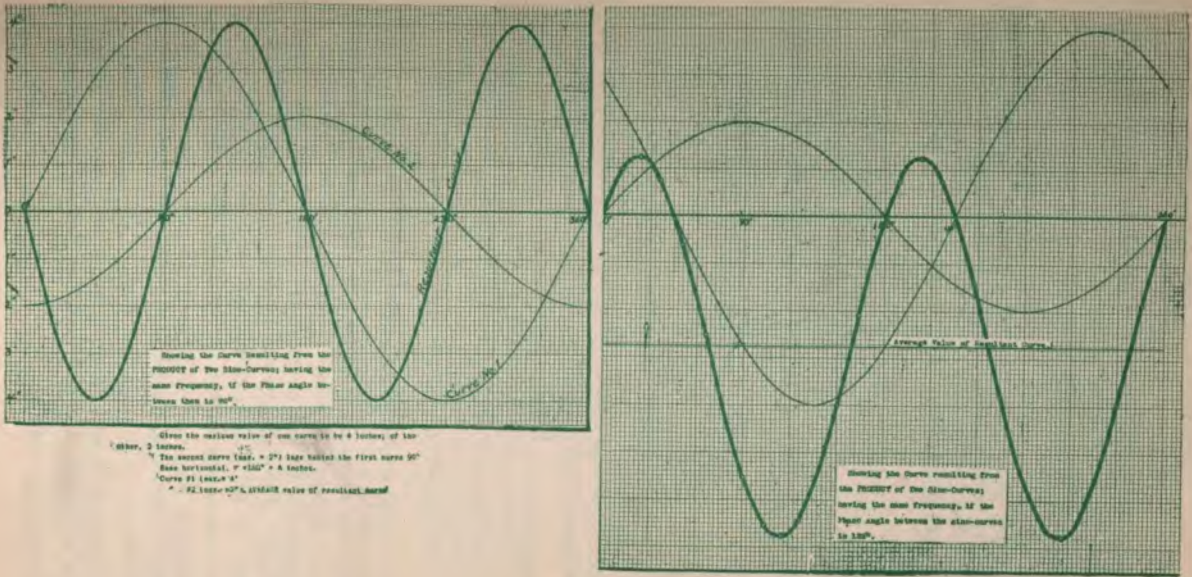
$$= \frac{\mathbf{E}_{\mathbf{max}}}{\sqrt{2}} \times \frac{\mathbf{I}_{\mathbf{max}}}{\sqrt{2}} \left(1 - \cos 2\,(2\pi f t)\right).$$


Average Value of Resultant Curve.
Resultant Curve
Curve No. I
Curve No. II
0°
90°
180°
270°
360°
Showing the curve resulting from the PRODUCT of two sine curves, when the sine curves are in phase.
[illegible]
Average Power = [illegible] watts, average Value of Resultant Curve.
Showing the Curve resulting from the PRODUCT of Two Sine-Curves; having the same frequency, if the Phase angle between the sine-curves is 15°
One inch = one volt, one ampere, one watt

a.

PLATE V.

b.



Showing the Curve Resulting from the PRODUCT of Two Sine-Curves; having the same frequency, if the Phase angle between them is 90°.

Given the maximum value of one curve to be 4 inches; of the other, 2 inches.

The second curve (max. = 2") lags behind the first curve 90°

Base horizontal, 0°–180° = 4 inches.

Curve #1 (max.= 4"

" – #2 (max.= 2"). AVERAGE value of resultant zero

Showing the Curve resulting from the PRODUCT of Two Sine-Curves; having the same frequency, if the Phase angle between the sine-curves is 135°.

a.

PLATE VI.

b.

PRODUCT OF TWO SINE-CURVES HAVING THE SAME FREQUENCY BUT WITH DIFFERENT PHASE DISPLACEMENTS.

Suppose the two curves are expressed by:

$$e = \mathbf{E}_{\max} \sin (2\pi f t + \Theta),$$
and $$i = \mathbf{I}_{\max} \sin (2\pi f t).$$

The current lags the pressure by an angle Θ.

Multiplying the two equations gives:

$$ei = \mathbf{E}_{\max}\, \mathbf{I}_{\max} \sin (2\pi f t + \Theta) \sin (2\pi f t)$$

Expanding the factor $\sin (2\pi f t + \Theta)$ according to (**6**) on page 13 gives:

$\sin (2\pi f t) \cos \Theta + \cos (2\pi f t) \sin \Theta$ and substituting this value gives:

$$ei = \mathbf{E}_{\max}\, \mathbf{I}_{\max} \sin^2 (2\pi f t) \cos \Theta + \sin (2\pi f t) \cos (2\pi f t) \sin \Theta.$$

Denoting ei by p and multiplying both members of the equation by dt gives:

$$p\,dt = \mathbf{E}_{\max}\, \mathbf{I}_{\max} \sin^2 (2\pi f t) \cos \Theta \, dt + \sin (2\pi f t) \cos (2\pi f t) \sin \Theta \, dt.$$

Integrating this expression between the limits of 0 and π, since the integral of the last term is zero,

gives; $$p \int_0^{\pi} dt = \frac{\mathbf{E}_{\max}\, \mathbf{I}_{\max}}{2} \cos \Theta = \mathbf{E}_{\text{eff}}\, \mathbf{I}_{\text{eff}} \cos \Theta.$$

The greater the value of the angle Θ separating the pressure and current the *less* the value of $\cos \Theta$ and therefore the less the value of the power. When $\Theta = 0°$ $\cos \Theta = 1$, and when $\Theta = 90°$, $\cos \Theta = 0$. Consult figure a, Plate VI, page 113.

EXAMPLE 30: Find the average value of the power, in watts, in a circuit to which is applied the pressure; $e = 1131.2 \sin (2\pi f t + 30°)$, **with the resulting current;** $i = 28.28 \sin (2\pi f t)$.

DATA.

$\mathbf{E}_{\max} = 1131.2;\ \mathbf{E}_{\text{eff}} = 800$ volts.
$\mathbf{I}_{\max} = 28.28;\ \mathbf{I}_{\text{eff}} = 20$ amperes.
$\Theta = 30°;\ \cos \Theta = \frac{\sqrt{3}}{2}$

Average power $= \mathbf{E}_{\text{eff}}\, \mathbf{I}_{\text{eff}} \cos \Theta$
$= 800 \times 20 \times \cos 30°$
$= 800 \times 20 \times \frac{\sqrt{3}}{2}$
$= \mathbf{8000 \times \sqrt{3}}$ **watts. Answer.**

PROBLEM 30: Find the average value of the power, in watts, in a circuit, if the applied pressure is; $e = 1131.2 \sin (2\pi f t + 60°)$ *and the current;* $i = 28.28 \sin (2\pi f t)$. **8000 watts** (very nearly). **Answer.**

Problem 30a: Find the average power, in watts, in a circuit, if the applied pressure is; $e = 1131.2 \sin (2\pi f t + 135°)$ and the current is: $i = 28.28 \sin (2\pi f t)$. Consult figure b, Plate VI, page 113,

INDUCTANCE OF COILS.

A *straight* conductor offers the same resistance to alternating as to direct currents. If however the conductor be wound into the form of a coil or solenoid, it is found to offer more *apparent* resistance to an alternating than to a direct-current, with the same value of applied pressure. So far as the direct-current values are concerned it makes no difference whether the conductor is straight or coiled; if the value of the applied direct pressure is the same, the current will be a constant. If a certain constant* alternating-pressure be applied to the terminals of a *straight* wire, the current in the wire is also found by applying Ohm's law; namely in symbols:

$\mathbf{I}=\frac{\mathbf{E}}{\mathbf{R}}$. If the wire is coiled up, with the same value of alternating-pressure, the value of the current in the wire is found to be very much *less* than when the wire was straight. The value of the *ohmic* resistance, **R**, of the wire is the same when the wire is straight as when it is coiled into a solenoid.

A given coil of wire in the shape of a solenoid having a length of 8½ inches, a diameter of central hole of 2 inches, consisting of 3000 turns of No. 16, B. & S. gauge, double cotton covered copper magnet wire, and a resistance, at 70° Fahrenheit of 10 ohms, was found by experiment to have a current of 10.83 amperes in it when a "direct-current" pressure of 108.3 volts was applied to its terminals; when however an "alternating-pressure" (effective) of 108.3 volts was applied to the terminals of the same coil a current of only 1.48 amperes (effective value) resulted. The ohmic resistance of the wire in the coil was 10 ohms in both cases. Some cause other than ordinary resistance was acting, when the alternating-pressure was applied, to reduce the value of the current. In order that 1.48 amperes be the value of current with an applied direct-pressure of 108.3 volts, the following relation must be true:

$$1.48=\frac{108.3}{10+\mathbf{R}_x};$$ from which the apparent added resistance was

$$\mathbf{R}_x=\frac{108.3-14.8}{1.48}$$

$$=56.41 \text{ ohms.}$$

The total apparent resistance offered to the alternating-current by the coil is the equivalent of 66.41 ohms, instead of 10 ohms. The *apparent* resistance is said to be 66.41 ohms in this case. If the wire had been *straight* and its resistance had been 66.41 ohms, then the current in it would have been 1.48 amperes for either a D. C. or an A. C. pressure of 108.3 volts at any frequency, applied to the terminals of the wire. Since the true ohmic resistance of the coil was 10 ohms, the reducing or the "*choking*" effect when the A. C. pressure was applied was due to a *counter* electromotive force. Such a counter electromotive force is called a counter electromotive force of self-induction or simply of *induction*. This counter electromotive force of induction is expressed by $e = 2\pi f \mathbf{L}\mathbf{I}$; f denoting the frequency in cycles per second of the applied A. C. pressure, and **L** denoting a coefficient, called the coefficient of inductance; depending in numerical value upon the strength of the magnetic field inside of the coil or solenoid.

---

*Meaning one whose maxima values are always the same.

The expression for the alternating-current, in amperes, in a coil or solenoid, having no iron near it, is

$$\mathbf{I} = \frac{\mathbf{E}}{\sqrt{\mathbf{R}^2 + (2\pi f \mathrm{L})^2}}$$; **E** denoting the alternating-pressure, in volts, (effective) applied to the terminals of the coil; **R** denoting the resistance in ohms, of the wire constituting the coil; f denoting the frequency, in cycles per second, of the applied alternating-pressure, and **L** denoting the coefficient of induction, expressed in henrys.

If the *maximum* value of the applied pressure is used in the above equation instead of the effective value, then the value of the current found by dividing the maximum value of the pressure by the value expressed by the denominator of the equation, will be the *maximum* value of the current. This is assuming that the applied pressure is a sine-pressure.

The expression $\sqrt{\mathbf{R}^2 + (2\pi f \mathbf{L})^2}$ is called the *impedance* of the coil and the expression $2\pi f \mathbf{L}$ is called the *reactance* of the coil. For convenience the impedance is sometimes denoted by **Z** and the reactance by **X**.

Then:

$$\mathbf{Z} = \sqrt{\mathbf{R}^2 + (2\pi f \mathbf{L})^2} = \sqrt{\mathbf{R}^2 + \mathbf{X}^2}, \text{ and } \mathbf{X}^2 = \mathbf{Z}^2 - \mathbf{R}^2;$$

$$\text{or } \mathbf{X} = \pm \sqrt{\mathbf{Z}^2 - \mathbf{R}^2}.$$

The numerical value of the coefficient of inductance **L** is readily obtained by experiment as indicated in example 31, page 117. The numerical values given in this example, and in problems 31a, 31b, 31c and 31d were indicated by the various instruments when connected as shown in figure 32.

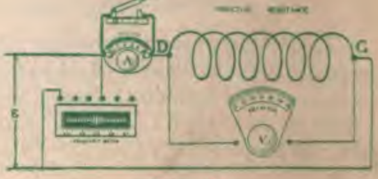


FIG. 32.

COEFFICIENT OF INDUCTANCE OF COILS.

EXAMPLE 31: If the effective value of the current in a coil or solenoid of copper wire having no iron or other magnetic material near it is found by measurement to be 1.48 amperes when the effective value of the pressure applied to its terminals is 108.3 volts at 60 cycles, compute the numerical value of the coefficient of inductance L of the coil, if its resistance is 10 ohms.

From the equation:

$$I = \frac{E}{\sqrt{R^2+(2\pi fL)^2}}$$

the expression for L is obtained as follows:

$$I^2 = \frac{E^2}{R^2+(2\pi fL)^2};$$

$$I^2R^2+(2\pi fLI)^2 = E^2;$$

$$2\pi fL = \sqrt{\frac{E^2-I^2R^2}{I^2}};$$

$$L = \frac{1}{2\pi f}\sqrt{\frac{E^2-I^2R^2}{I^2}}.$$

OBSERVED DATA.

$E = 108.3$ volts.

$f = 60 \sim$.

$I = 1.48$ amperes. (effective value)

$R = 10$ ohms.

CALCULATED DATA.

$E^2 = 11728.89$

$I^2 = 2.19$

$R^2 = 100$

$\frac{1}{2\pi f} = .00265$ (Consult table, page 209.)

Substituting the proper numerical values in this equation gives:

$$L = .00265\sqrt{\frac{11728.89-219}{2.19}} = .00265\sqrt{5255.65}.$$

$$= .00265\times 72.49 = 0.19 \text{ henry.}$$ **Answer.**

PROBLEM 31: Given a coil of insulated copper wire, having a resistance of 10 ohms, with no magnetic material near it. If a pressure of 104 volts at 60 cycles applied to the terminals of the coil produces a current of $2\frac{4}{10}$ amperes in it, compute the numerical value of the coefficient of inductance, L, of the coil. **0.1118 henry. Answer.**

Problem 31a: If a pressure of 82.5 volts at 60 cycles, applied to the terminals of a coil of copper wire having no magnetic material near it, produces a current of 1.17 amperes in the coil, compute the numerical value of the coefficient of inductance of the coil.

Problem 31b: Given a coil of wire having no iron core, whose resistance is 7,000 ohms. Compute the coefficient of inductance of the coil, if a pressure of 104 volts at 60 cycles, when applied to the terminals of the coil produces a current of $\frac{13}{1000}$ amperes in it.

L = 10.26 henrys. Answer.

Problem 31c: Given a coil of wire, having no iron core, whose resistance is 10 ohms, compute the coefficient of inductance of the coil, if a pressure of 110 volts (effective) at 60 cycles, when applied to the terminals of the coil produces a current of $\frac{41}{100}$ amperes (maximum) in it. *Effective* pressure is given and *maximum* current required.

Problem 31d: Given a coil of copper wire, with no iron core, whose resistance is 10 ohms. Compute the numerical value of the coefficient of inductance of the coil, if a pressure of 54.15 volts at 60 cycles, when applied to the terminals of the coil produces a current of 0.74 amperes in it. Compare the answer to this problem with EXAMPLE 31, page 119.

The coefficient of inductance of a coil *having no iron within or near it*, is a constant regardless of the strength of current in the wire of the coil.

The introduction of an iron core into a coil greatly increases the inductance for all current values, and in addition the coefficient of inductance becomes variable. The variability of the inductance now depends upon the magnetic permeability of the iron constituting the core.

It is not possible to *compute* the numerical value of the coefficient **L** when iron is present, unless the magnetic permeability, (denoted by μ) is known for each and every value of current in the wire constituting the coil.

A number of equations have been derived from theoretical considerations and from experiment, for calculating the coefficient of inductance of any given coil having no iron near it, in terms of the *dimensions* of the coil. Two such equations will be considered.

CALCULATION OF THE COEFFICIENT OF INDUCTANCE OF A COIL.

EXAMPLE 32: Compute the coefficient of inductance L, of a coil of insulated copper wire; the length of the coil being 21.6 centimeters (8½ inches) **over all, its diameter being 11.74 centimeters** (4⅝ inches) **and having a central hole 5.08 centimeters** (2 inches) **in diameter, if the number of turns of wire is 3000.**

The value of **L** in this case will be computed from two different equations and the results compared.

The dimensions here given were measured from an actual coil wound with double cotton covered, No. 16, B. & S. copper magnet wire.

Using the equation:*

$$L = \frac{4\pi^2 a^2 N^2}{b+c+r} XY \times \frac{1}{10^9},$$

in which **a** denotes the mean radius of the windings.

r denotes the radius of the coil.

b denotes the axial length of the coil.

c denotes the thickness of the winding.

N denotes the total number of turns in the windings.

$$X = \frac{5b+6c+r}{5b+5c+\frac{7}{10}r};$$

and $$Y = \tfrac{1}{2}\log_{10}\left(100 + \frac{14r}{2b+3c}\right).$$

GIVEN AND COMPUTED DATA.

$b = 21.6$ centimeters

$5b = 108$

$N = 3000$; $N^2 = 9{,}000{,}000$.

$a = \frac{2.54+5.87}{2} = 4.2$ cms.

$c = 3.33$ cms.

$3c = 9.99$

$5c = 16.65$

$6c = 19.98$

$r = 5.87$ cms.

$\frac{7}{10}r = 4.109.$

$14r = 57.526$

$\pi = 3.1416.$

$4\pi^2 = 39.4784.$ (See page 213.)

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*Found in *Inductance of Coils* by Morgan Brooks and H. M. Turner, *University of Illinois Bulletin* Vol. IX, No. 10. A very interesting and valuable discussion.

Substituting the proper numerical values in the preceding equations give:

$$X = \frac{108+19.98+5.87}{108+16.65+4.109}$$

$$= 1.039;$$

$$Y = \tfrac{1}{2}\log_{10}(100+1.081) = \tfrac{1}{2}(2.0046652)$$

$$= 1.00233;$$

and $$\frac{4\pi^2a^2N^2}{b+c+r} \times \frac{1}{10^9} = \frac{39.4784 \times 17.64 \times 9\,000\,000}{21.6+3.33+5.87} \times \frac{1}{10^9}$$

$$= 0.203493; \text{ therefore}$$

$$L = 0.203493 \times 1.039 \times 1.00233$$

$$= \mathbf{0.2125\ henry.}$$

The value of **L** for the above coil obtained experimentally, according to the method outlined on page 116 was found to be 0.18.

In obtaining the numerical value of **L** according to this method, it may be seen that a slight error in observation or in the indications of the voltmeter and the ammeter affects the calculated results, since the values of the pressure and current are *squared*. This is evident by a study of example 31, page 117.

EXAMPLE 32a: Compute the numerical value of the coefficient of inductance L, for a coil of copper wire consisting of 3000 turns of insulated wire having a resistance of 10 ohms at 70°F if the length of the coil is 8½ inches, with a 2 inch diameter central hole through it.

The value of **L**, in henrys, is expressed* by;

$$L = \frac{4\pi n^2 \mu A}{b} \times 10^{-9}$$

$$= \frac{4\pi n^2 \mu A}{1\,000\,000\,000\, b}.$$

DATA.

$4\pi = 12.56636.$

$n = 3000,\ n^2 = 9\,000\,000.$

$\mu = 1$ for air and for any non-magnetic material.

$A = \pi \times \overline{2.54}^2 = 20.27$ sq. cms.

$b = 8.5$ inches; $= 21.59$ cms.

$= 21.6$ cms (very nearly)

$\frac{A}{b} = 0.938.$

Substituting in the above equation the given numerical values gives:

$$L = \frac{12.56636 \times 9\,000\,000 \times 0.938}{1\,000\,000\,000} = \mathbf{0.106\ henry.} \quad \textbf{Answer.}$$

---

*See *Alternating-Current Machines* by Shelden, Mason, and Hausmann, page 30.

It may be noted that this equation requires fewer data than the previous equation, and that errors in measurements are not magnified in the process of computation.

Problem 32a: Compute the coefficient of inductance of a two layer coil, using the following equation:

$$\mathbf{L} = 4\pi^2 n^2 m \left\{ \frac{2\mathrm{a}^4 = a^2 l^2}{\sqrt{4a^2+l^2}} - \frac{8a^3}{3\pi} \right\} + 8\pi^2 n^2 \times$$

$$\Big\{ [\,(m-1)\ \ a_{,}^2 + (m-2)\ \ a_{,,}^2 + \ldots .]\,[\sqrt{a_{,}^2+l^2} - \tfrac{7}{8} a_{,}] +$$

$$\tfrac{1}{2}[m(m-1)\,a_{,}^2 + (m-1)\,(m-2)\,a_{,,}^2 + (m-2)\,(m-3)\,a_{,,,}^2 + \ldots .] \times$$

$$\left(\frac{a_{,}\delta a}{\sqrt{a_{,}^2+l^2}} - \delta a \right] \Big\}$$; in which m denotes the number of layers;

a the mean radius of the windings; $a_{,}$, $a_{,,}$, $a_{,,,}$, etc., the mean radii of the several layers; l the length of the solenoid, in centimeters; n the number of turns per centimeter length; δa the radial distance, in centimeters, between any two consecutive layers.

For the coil mentioned above, $m = 2$; $a = 2.669$ cms.; $n = 6.06$; $\delta a = 0.129$ cm., and $l = 21.6$ cms.

This gives the inductance, in henrys, of the two inside layers of the coil mentioned in example 32, page 119.

INDUCTANCE OF TRANSMISSION LINES.

Whenever energy is transmitted from one locality to another by electrical transmission lines employing alternating-currents, "*inductance*" of the transmission lines plays a most important role in the "efficiency of operation" of the lines.

The greater the inductance of the lines the less the power factor of the lines and the greater the amount of copper needed to deliver any given amount of power. This fact may be better comprehended by consulting pages 50 and 53, Vol. II, Examples in Alternating-Currents. For the present it will be only advisable to consider the factors entering into "*line inductance*", their relation with each other and their individual influence on the numerical value of the coefficient of inductance of transmission lines under ordinary conditions of installation.

EXAMPLE 33: Compute the inductance L of a transmission line 100 miles long, consisting of two No. 1, B. and S. gauge copper wires, strung so that the distance between their centers is 2 feet.

The equation expressing the inductance of two parallel wires is:

$$\mathbf{L} = 4l\left\{\left[\log_e \frac{d}{r}\right] + \frac{\mu}{4}\right\} \times \frac{1}{10^9}$$

DATA.

$l = 100$ miles $= 16093440$ centimeters.
$d = 24$ inches $= 60.96$ centimeters.
$r = 0.1446$ inch $= 0.3674$ cms.
$\mu = 1$ (for air)
$\frac{d}{r} = 165.9$

In which **L** denotes inductance, in henrys; $\log_e$ of $165.9 = 5.111988$ (from tables) l denotes the length of the *line* (one wire) in centimeters; d the distance between the wires; r the radius of the wire; μ the magnetic permeability of the medium between the wires; (for air $\mu = 1$); $\log_e$ denotes natural logarithms.

To convert common logarithms (base 10) into natural logarithms (base 2.7182818+) multiply the common logarithm by 2.3022585; or divide the common logarithm by the modulus 0.4342945 which is the reciprocal of 2.3022585.

The ratio $\frac{d}{r}$ is the same numerically, whether the numerical values of d and r are expressed in inches or centimeters.

Substituting the various numerical values in the fundamental equation gives:

$$\mathbf{L} = 64373760\,[5.111988 + 0.25] \times \frac{1}{10^9} = 0.28 \text{ henry.}$$

PROBLEM 33: Find the coefficient of inductance **L**, *of an alternating-current transmission line consisting of two* No. 1, **B.** & **S.** *gauge copper wires, strung so the distance between their centers is 48 inches. The distance being 10 miles.* **.038 henry. Answer.**

$$\log_e \frac{d}{r} = 5.761244.$$

Problem 33a: Find the coefficient of inductance **L**, of an A. C. transmission line consisting of two No. **00** **B.** & **S.** gauge aluminum wires, each 10 miles long, strung so that the distance between their centers is 48 inches.

As the length of the line increases, with any given spacing, the inductance coefficient increases; as the spacing increases with any given length of line, the coefficient *increases.* As the size of the line wire increases for any given length, and spacing, the line inductance *decreases.*

EXAMPLE 34: Find the numerical value of the effective current, in amperes, in a coil having a coefficient of inductance $L = 0.112$ henry, and a resistance of 10 ohms; if an alternating-sine-wave pressure, having a maximum value of 147.07 volts at 60 cycles, is applied to the terminals of the coil.

The expression for the current in the coil is:

$$I = \frac{E}{\sqrt{R^2 + (2\pi f L)^2}}.$$

DATA.

$E_{max} = 147.07$ volts.

$E_{eff} = 104$ volts.

$(147.07 \times \sqrt{\tfrac{1}{2}}) = (147.07 \times .707) = 104$

$f = 60$ cycles per second.

$R = 10$ ohms.

$2\pi f = 377$

The current in the coil for a frequency of 60 cycles will be:

$$I = \frac{104}{\sqrt{100 + (377 \times 0.112)^2}}$$

$$= \frac{104}{\sqrt{100 + 1782}} = \frac{104}{\sqrt{1882}} = \frac{104}{43},$$

$=$ **2.4 AMPERES;** (effective value). **Answer.**

PROBLEM 34: Find the current in the coil mentioned in EXAMPLE 34 if the frequency of the pressure is changed to $25\sim$, *all other conditions remaining as stated.* **5.2 Amperes;** (effective). **Answer.**

Problem 34a: Plot a curve, to scale, showing the variation in the current in the coil mentioned in EXAMPLE 34 due to variation in the *frequency* of the applied pressure; all other values remaining constant as stated in the example. Take values of frequency as follows: 15, 25, 60, 80, 100, 125. Plot values of frequency horizontally as abscissae, and values of current, in amperes, vertically as ordinates.

Problem 34b: If the reactance and the resistance of a coil with no iron near, are equal to each other when the coil is connected in a circuit where the frequency of the A. C. pressure is 15 cycles, what will their relative values be when the same coil is connected with a circuit where the A. C. pressure has the same numerical value at 60 cycles? The frequencies are as 1 to 4. Consult example 36, page 127 and page 115a.

INDUCTANCE OF FIELD COILS.

EXAMPLE 35: Find the inductance of the field coils of a 20 **pole alternator, if the** 20 **coils are connected together in series; each coil having** 616 **turns, if a field current of** 6.95 **amperes produces** 6,400,000 **maxwells** (6.4 megalines) **per pole.**

For each turn;

$$L=\frac{\Phi}{i}$$

DATA.

$\Phi=6{,}400{,}000$ maxwells.

$N=$

$i=\frac{6.95}{10}=0.695$ abamperes.

For 616 turns:

$$L=616\ \frac{\Phi}{i}=\frac{616\times6400000}{0.695}$$

$=$**5.672 henrys, per coil. Answer.**

For 20 coils the total inductance will be $5.672\times20=113.44$ henrys.

PROBLEM 35: Compute the coefficient of inductance, **L**, *of a two-pole alternator having 3750 ampere-turns per field coil and a magnetic flux of 4.8 mega-maxwells, when the exciting current* (current in the field coils) *is* 1 *ampere.* **45 henrys per coil, 90 henrys for both coils. Answer.**

Problem 35a: Find the field winding inductance of a four-pole alternator, having 5000 ampere-turns per pole, and a total flux of 5,000,000 maxwells when the exciting current is 5 amperes.

POWER FACTOR.

Whenever an alternating-pressure is applied to the terminals of a coil such as is mentioned in example 31, page 117, and in example 32, page 119, the pressure may be considered to be divided into *components*, as illustrated in figure 33, page 129.

One component of the applied pressure is effective in producing the so-called ohmic current; which is useful in producing the desired useful power effects. If the applied pressure is a sine-wave pressure, and there is no iron in or near the coil, the *ohmic pressure component* will be a sine-wave, the current will be a sine-wave current, and the current will be in phase with the ohmic pressure component.

The ohmic pressure component itself cannot be in phase with the applied pressure, and hence the so-called ohmic current, or the useful power component cannot be in phase with the applied pressure.

In figure 33, page 129, the ohmic pressure component is denoted by **RI** and the applied pressure by **E**.

Here the **RI** ohmic component lags behind the applied pressure by an angle of **45°**.

Whenever the ohmic current lags the applied pressure, **the cosine of the angle of lag is called the power factor.**

This is however only true for sine-curve pressures and components, and their resulting sine-curve currents and components.

In figure 33, the angle of lag being 45°, the cosine is 0.70711, and the power factor is 0.707 (approx.) or $(0.707 \times 100) = 70.7\%$.

If the angle of lag is 0°, the cosine of 0° being 1, the power factor would be unity; or 100%.

Any circuit having inductance (meaning one whose coefficient of inductance has any numerical value other than zero) has a power factor whose numerical value is other than unity.

Alternating-current motors and transformers have power factors other than unity, under ordinary operative conditions.

The ordinary liquid resistances or liquid rheostats have unity power factor.

There is no definite power factor in cases where the alternating-pressures and currents are other than sine-curve in shape.

Power factor is not a fixed **constant value** for any given machine or circuit. In a motor and transformer it varies with the load output. The same is true of a *circuit* supplying power for several motors. The power factor of a given circuit, with one motor operating at a steady load output, is a definite value; when another motor is connected with the circuit, the power factor changes in value.

In a two-phase circuit, one circuit, or one phase, may have a certain power factor, while the other circuit or the second phase may have a different power factor.

The three phases of a three-phase circuit may have different power factors.

The power factor of a poly-phase circuit is an indefinite quantity.

Power factor will be discussed further, in VOL. II.

EXAMPLE 36: If a harmonic (*sine-wave*) pressure of 155.5 volts, maximum value, when applied to a circuit, produces a current of 10 amperes, (*effective value*) which lags the applied pressure an angle of 45°, find the resistance and the reactance of the circuit.

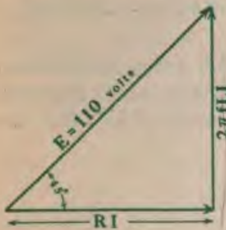

FIG. 33.

DATA.
$E_{max}=155.5$ volts.
$E_{eff}=110$ volts.
$I=10$ amperes. (eff)
$\phi=45°$.
$\tan\phi=1$
$\cos\phi=.707$
$\sin\phi=.707$.

From:—

$$I=\frac{E}{\sqrt{R^2+(2\pi fL)^2}},$$

$$\text{impedance }=\sqrt{R^2+(2\pi fL)^2}=\frac{E}{I}=\frac{110}{10}=11=Z.$$

In the triangle shown in figure 33, which represents the vector relations of the applied pressure and its reactive and resistance components, the side adjacent $=$**RI**; the hypotenuse $=110$, and the side opposite $=$ $2\pi f$**LI**. From trigonometrical relations:

$110\times\cos 45°=$ **RI** $=77.777$. Therefore the resistance

$$R=\frac{110\times.707}{10}=\textbf{7.77.OHMS.}\qquad\textbf{Answer.}$$

The reactive **E. M. F.** $=2\pi f$**LI**, and from trigonometrical relations:

110 sin **45°** $=2\pi f$**LI**; or the **reactance;**

$$X=2\pi fL=\frac{110\times0.707}{10}=7.77\text{ (expressed in ohms.)}$$

If $2\pi f=157$ in the above example, find the value of L and of the frequency.

From $2\pi fL=7.77$, if $2\pi f=157$, $f=25$ cycles. (See table page 208.)

$L=\frac{7.77}{157}=.04949$ **henry.**

PROBLEM 36: If a harmonic (sine-wave) *pressure of 155.5 volts* **maximum,** *when applied to a circuit, produces a sine-current of 10 amperes, which lags the applied pressure by an angle of 30°, find the resistance and the reactance of the circuit.* (Consult page 19.)

Resistance = 9.52 ohms. Reactance = 5.5 ohms. Answer.

Problem 36a: If a harmonic pressure of 155.5 volts, **maximum,** when applied to a given circuit, produces a sine-current of 10 amperes, effective value, lagging the applied pressure by an angle of 15°, find the resistance and the reactance of the circuit.

Find the value of **L** if $2\pi f = 377$.

Find the value of **L** if $2\pi f = 157$.

INDUCTIVE IMPEDANCES IN SERIES.

Suppose two coils of insulated copper wire, having different inductances, without iron cores and with no magnetic material near them are connected together in *series*, and an alternating-pressure at a certain frequency is applied to the terminals of the arrangement. It is desired to know the numerical values of the various pressure components; their phase relations with either the applied pressure or with some *current component*, and also the magnitude and phase of the current components.

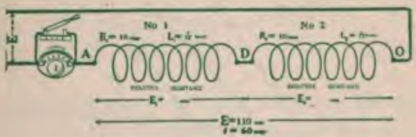

FIG. 34.

The arrangement is denoted by figure 34.

Let $\mathbf{R}_{,}$ denote the resistance of coil No. 1; expressed in ohms.

Let $\mathbf{L}_{,}$ denote the coefficient of inductance of No. 1, expressed in henrys.

Let $\mathbf{R}_{,,}$ denote the resistance of coil No. 2; expressed in ohms.

Let $\mathbf{L}_{,,}$ denote the coefficient of inductance of No. 2; expressed in henrys.

Let f denote the frequency of the applied pressure.

The *impedance* of coil No. 1 may be expressed by:

$$\mathbf{Z}_{,}=\sqrt{\mathbf{R}_{,}^{2}+(2\pi f\mathbf{L}_{,})^{2}}=\sqrt{\mathbf{R}_{,}^{2}+\mathbf{X}_{,}^{2}}.$$

The *impedance* of coil No. 2 may be expressed by:

$$\mathbf{Z}_{,,}=\sqrt{\mathbf{R}_{,,}^{2}+(2\pi f\mathbf{L}_{,,})^{2}}=\sqrt{\mathbf{R}_{,,}^{2}+\mathbf{X}_{,,}^{2}}.$$

The vector diagram, figure 35, page 132, shows the relation of the current and the various pressure components.

Since whatever current (denoted by **I**) flows in one coil must flow in the other coil, the coils being in series, it will be convenient to employ the *ohmic pressure-component* (which is in line with the current) as the horizontal datum. In figure 35 the *ohmic pressure-component*, ($R_{,}$ I) of coil No. 1, is denoted by the vector **DF**. The *reactive-pressure-component* ($2\pi f L_{,} I$) of the same coil is denoted by vector **FA**. The *resultant* of these two pressure-components is $E_{,}$ denoted by vector **DA**.

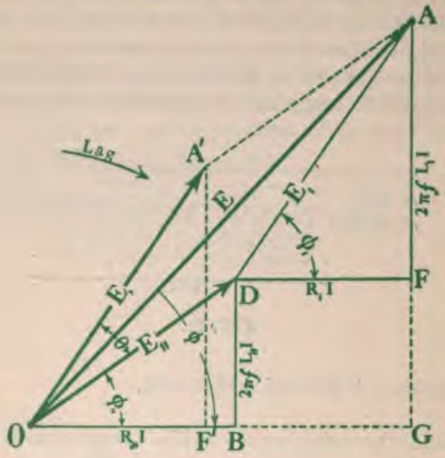

FIG. 35.

The angle of lag of the ohmic-pressure, ($R_{,}$ I), and likewise of the current-component I, behind $E_{,}$, is denoted by the angle $\phi_{,}$.

$$\text{Then:}\quad I = \frac{E_{,}}{\sqrt{R_{,}^2 + (2\pi f L_{,})^2}} = \frac{E_{,}}{Z_{,}}\ ;\ \tan\phi_{,} = \frac{2\pi f L_{,}}{R_{,}}\ .$$

$$\cos\phi_{,} = \frac{R_{,}\, I}{E_{,}}\ ;\ \sin\phi_{,} = \frac{2\pi f L_{,}}{E_{,}}\ .$$

Referring to the same figure, the ohmic pressure-component, (R'' I), of coil No. 2, is denoted by the vector **OB**; the reactive pressure-component, ($2\pi f L_{,,} I$), by vector **BD**; the resultant $E_{,,}$ of these two components, by vector **OD**, and the angle of lag of ($R_{,,}$ I) behind $E_{,,}$, by $\phi_{,,}$.

Then:

$$I = \frac{E_{,,}}{\sqrt{R_{,,}^2 + (2\pi f L_{,,})^2}} = \frac{E_{,,}}{Z_{,,}}\ ;\ \tan\phi_{,,} = \frac{2\pi f L_{,,}}{R_{,,}}\ ;$$

$$\cos\phi_{,,} = \frac{R_{,,} I_{,,}}{E_{,,}}\ ;\ \sin\phi_{,,} = \frac{2\pi f L_{,,}}{E_{,,}}\ .$$

The resultant of $\mathbf{E}_{,}$ and $\mathbf{E}_{,,}$ is $\mathbf{E}$; denoted by vector **OA**. The triangle **DAF** might have been constructed as indicated by **OA′ F′**. Then from the expression for the resultant of two forces:

$\mathbf{E} = \sqrt{\mathbf{E}_{,}^2 + \mathbf{E}_{,,}^2 + 2\,\mathbf{E}_{,}\mathbf{E}_{,,}\cos\theta}$; in which θ denotes the angle between the two *pressure components* $\mathbf{E}_{,}$ and $\mathbf{E}_{,,}$. It should be noted that $\theta = \phi_{,} - \phi_{,,}$.

If the angle between the current $\mathbf{I}$ and the resultant or applied pressure $\mathbf{E}$ is denoted by ϕ, then;

$$\tan\phi = \frac{2\pi f \mathbf{L}_{,} + 2\pi f \mathbf{L}_{,,}}{\mathbf{R}_{,} + \mathbf{R}_{,,}}.$$

One of the coils represented in figure 34 might be wound noninductively, or one of the series portions might consist of a noninductive resistance. If this should be section No. 2 then $\mathbf{E}_{,,}$ would be horizontal in figure 35 and in line with $\mathbf{R}_{,}\,\mathbf{I}_{,}$ (**D F**).

If the two coils are exactly similar; having the same resistance and the same inductance, then $\phi_{,} = \phi_{,,}$; $\mathbf{E}$, $\mathbf{E}_{,}$ and $\mathbf{E}_{,,}$ will be in line, and $\mathbf{E}_{,} + \mathbf{E}_{,,} = \mathbf{E}$.

It is evidently impossible for $\mathbf{E}_{,}$ and $\mathbf{E}_{,,}$ to be at right angles with each other. The angle **ODA** between $\mathbf{E}_{,}$ and $\mathbf{E}_{,,}$ approaches 90° if the resistance of one coil is large and its inductance is small, while the resistance of the other coil is small and its inductance is large.

EXAMPLE 37: If a coil having a resistance of 10 ohms and an inductance $L_{\prime}=\frac{3}{10}$ henry, is connected in series with another coil having a resistance of 100 ohms and an inductance $L_{\prime\prime}=\frac{1}{10}$ henry, and a pressure of 200 volts, maximum value is applied to the terminals of the arrangement, find the effective value of the current, in amperes, in the circuit, and the numerical value and phase relation of the pressure between the terminals of each coil. The frequency being 60 ∼.

The effective current in the circuit is:

$$I_{eff}=\frac{E_{eff}}{\sqrt{(R_{\prime}+R_{\prime\prime})^2+(2\pi f L_{\prime}+2\pi f L_{\prime\prime})^2}}\ {}^{*}$$

$$=\frac{141.4}{\sqrt{110^2+150.8^2}}$$

$$=\frac{141.4}{186.6}$$

$=0.75$ **ampere. Answer.**

GIVEN DATA.

$E_{max}=200$ volts.
$f=60$ cycles.
$L_{\prime}=\frac{3}{10}$ henry.
$R_{\prime}=10$ ohms.
$L_{\prime\prime}=\frac{1}{10}$ henry.
$R_{\prime\prime}=100$ ohms.

COMPUTED DATA.

$E_{eff}=0.707\times200=141.4$ volts.
$2\pi f=377.$ (See page 208.)
$2\pi f L_{\prime}=113.1.$
$2\pi f L_{\prime\prime}=37.7.$

O

If the current is assumed as the horizontal datum, the angle ϕ between the applied pressure of 200 volts, maximum, and the maximum value of the current is found from;

$$\tan\phi=\frac{2\pi f L_{\prime}+2\pi f L_{\prime\prime}}{R_{\prime}+R_{\prime\prime}}=\frac{150.8}{110}=1.371;\ \text{from which}$$

$$\phi=+53^\circ\ 53', \text{ and is an angle of lag.}$$

The numerical value of the pressure (effective) between the terminals of one coil is $I_{eff}\sqrt{R_{\prime}^2+(2\pi f L_{\prime})^2}=85.05$ volts.

The angle between this pressure and the current is found from;

$$\tan\phi_{\prime}=\frac{2\pi f L_{\prime}}{10}=11.31;\ \text{from which is found}$$

$$\phi_{\prime}=+84^\circ\ 57';\ \text{therefore the current lags this pressure.}$$

---

*To lessen the labor in computation the denominator of this equation might be expressed as follows:

$$\sqrt{(R_{\prime}+R_{\prime\prime})^2+4\pi^2 f^2\,(L_{\prime}+L_{\prime\prime})^2}$$

The value of $4\pi^2 f^2=(2\pi f)^2$ may be obtained from the table on page 208.

The numerical value of the pressure (effective) between the terminals of the other coil is $\mathbf{I}_{\text{eff}}\sqrt{\mathrm{R}_{,,}{}^2+(2\pi f\mathbf{L}_{,,})^2}=80.14$ volts.

The angle between this pressure and the current is found from;

$$\tan\,\phi_{,,}=\frac{2\pi f\mathbf{L}_{,,}}{\mathrm{R}_{,,}}=0.377 \text{ from which is found}$$

$\phi_{,,}=20^\circ\ 40'$ and since the tangent is *positive*, the current *lags* the pressure of 80.14 volts.

In a case of this nature, $\phi_{,}-\phi_{,,}$ is always less than ϕ, and with any arrangement of inductive and noninductive resistances in series, the angle of lag can never be as great as 90°. (Consult figure 35, page 132.)

The *arithmetical* sum of the two pressures; $85.05+80.14=165.19$.

PROBLEM 37: If a coil having a resistance of 10 ohms and an inductance $\mathbf{L}_{,}=\frac{3}{10}$ *henry, is connected in series with another coil having a resistance of 10 ohms and a coefficient* $\mathbf{L}_{,,}=\frac{3}{10}$ *henry, and a pressure of 2000 volts, effective value, is applied to the terminals of the arrangement, find the effective value of the current, in amperes, in the arrangement, together with the numerical value and phase relation of the pressure across each coil. The frequency of the applied pressure being 60 cycles.*

8.81 amperes; $\mathbf{E}_{,}=\mathbf{E}_{,,}=1000$ volts; $\phi=\phi_{,}=\phi_{,,}=84^\circ 57'$. Answer.

Problem 37a: Plot a vector diagram, to scale, showing the relation of the applied pressure and the various component pressures in example 37, page 134, letting a length of 1 inch denote 20 volts. Assume all ohmic pressure-components to be horizontal.

Problem 37b: Find the effective value of the current in a circuit consisting of two inductive coils connected together in series, if one has a resistance $\mathbf{R}_{,}$ of 2 ohms and a coefficient of inductance $\mathbf{L}_{,}$ of $\frac{1}{100}$ henry; the other a resistance $\mathbf{R}_{,,}$ of 5 ohms and a coefficient of inductance $\mathbf{L}_{,,}$ of $\frac{1}{1000}$ henry, if the maximum value of a sine-wave pressure applied to the terminals of the arrangement is 311 volts at 15 cycles.

Find the numerical value and the phase relation of each coil pressure, referred to the current.

Effective applied pressure = 220 volts.

31.1 amperes; effective. $\mathbf{E_{\prime\prime} = 68.73}$ **volts.** $\mathbf{E_{\prime\prime} = 155.52}$ **volts.**
Tan $\phi_{\prime} = \mathbf{0.4712}$ $\phi_{\prime} = \mathbf{25^\circ\, 14'}$ Tan $\phi_{\prime\prime} = \mathbf{.018848}$ $\phi_{\prime\prime} = \mathbf{1^\circ\ 5'}$.
$\phi = \mathbf{8^\circ\ 25'}$

It should be carefully noted that a change in the frequency of the pressure applied to any coil or any arrangement of coils not only varies the numerical value of the resulting current but also changes the phase relation of the current with the applied pressure, assuming that the *numerical* value of the applied pressure, in volts, remains constant.

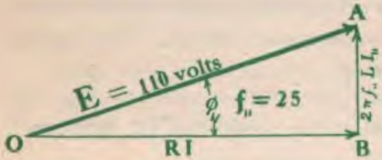

FIG. 36.

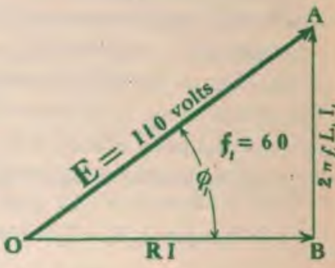

FIG. 37.

This may be shown graphically by means of vector diagrams such as figures 36 and 37. In both cases illustrated by these diagrams, the applied pressure **E** was the same; 110 volts, while in one case the frequency was 25 cycles, in the other it was 60 cycles. **R** was 100 ohms and **L** was 0.206 henry in each case.

For 60 cycles; $2\pi f'\mathbf{L} = 77.7$; $\tan\phi_{\prime} = \frac{77.7}{100} = 0.777$;
$\phi_{\prime} = 37^\circ\, 51'$; $\mathbf{I}_{\prime} = 0.868$ ampere, and $\mathbf{RI}_{\prime} = 86.8$ volts.

For 25 cycles; $2\pi f''\mathbf{L} = 67.44$; $\tan\phi_{\prime\prime} = \frac{67.44}{100} = 0.6744$.
$\phi_{\prime\prime} = 17^\circ 55'$; $\mathbf{I}_{\prime\prime} = 1.06$ ampere, and $\mathbf{RI}_{\prime\prime} = 106$ volts.

Problem 37c: A non-inductive resistance of 65 ohms, is connected in series with an inductive resistance of 5 ohms and $\frac{2}{10}$ henry, and a sine-pressure of 110 volts, (effective) at 60 cycles is applied to the terminals of the arrangement. Find the effective value of the current, in amperes, and its phase angle with reference to the applied pressure.

$I = 1.069$ amperes; $\phi = 47° 8'$; Answer.

Problem 37d: A non-inductive resistance of 10 ohms, is connected in series with an inductive coil having a resistance of 10 ohms and an inductance of 0.1118 henry. If a sine-pressure of 128 volts, (effective value) at 60 cycles is applied to the terminals of the arrangement, find the effective value of the current, in amperes, and its phase angle with reference to the applied pressure.

Problem 37e: Plot a vector diagram, to scale, showing the relative magnitudes and phase relations of the various pressures of problems 37c and 37d, letting 1 inch denote 20 volts.

The relation between impedance (see page 144) and admittance; between conductance and susceptance, and between resistance and reactance may perhaps be explained by the following:

Suppose figure 38 denotes an *impedance triangle*, **OBA**, properly drawn to scale.

This is a right-triangle (right angle at **B**) having its hypotenuse $\mathbf{OA} = \sqrt{R^2 + (2\pi f L)^2} =$ the *impedance* of a circuit, having a certain resistance **R**, coefficient of inductance **L** and an infinite capacity; (see page 152). The side $\mathbf{OB} = \mathbf{R}$, = the *resistance*; (expressed in ohms) the side $\mathbf{BA} = 2\pi f L =$ the *reactance*, and the angle between the resistance and the impedance $= \phi$. Tan $\phi = \frac{1.5}{2} = 0.75$; $\phi = 36°52'$; $\cos \phi = 0.80003$; $\sin \phi = 0.59995$.

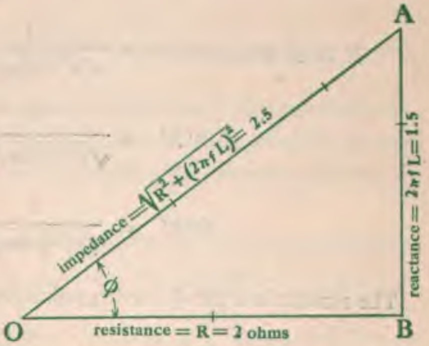
A
impedance $= \sqrt{R^2 + (2\pi f L)^2} = 2.5$
reactance $= 2\pi f L = 1.5$
ϕ
O
resistance = R = 2 ohms
B

FIG. 38.

Effective applied pressure

31.3 amperes; effective. $E_{a,r}$=

Tan ϕ_r=0.4712 ϕ_r=25°14' Tan

ϕ=5° 25'

[illegible] carefully noted that a

[illegible] to any coil or any

[illegible] of the resulting current

[illegible] with the applied pressure

[illegible] in volts, remain

It might be assumed that:

$$\sqrt{R^2+(2\pi f L)^2} = 2.5.$$

$$2\pi f L = 1.5$$

$$\text{and } R = 2$$

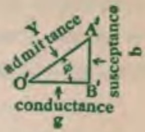

FIG. 39.

Next suppose the triangle **O′B′A′** figure 39 is so drawn that **O′A′** $= \frac{1}{2.5}$ (the reciprocal of $\sqrt{R^2+(2\pi f L)^2}$ and so that the angle **B′O′A′** (the angle at **O′**) is equal to ϕ as in figure 38. It will be impossible to properly show on a page of this size the true relative lengths of an impedance of 100 and its reciprocal $\frac{1}{100}$.

Admittance *is defined as the reciprocal of the impedance.* In figure 39 **O′ A′** denotes the *admittance*; the symbol **Y** is usually adopted to designate the admittance.

While the length of **O′ A′** figure 39 denotes the reciprocal of **OA** in figure 38, it does not follow that **O′ B′** is the reciprocal of **OB**; nor that **B′A′** is the reciprocal of **BA**.

If **O′ A′** in figure 39 is $= \dfrac{1}{\sqrt{R^2+(2\pi f L)^2}}$ then it is true that

$$B'A' = \frac{1}{\sqrt{R^2+(2\pi f L)^2}} \sin\phi, \text{ and}$$

$$O'B' = \frac{1}{\sqrt{R^2+(2\pi f L)^2}} \cos\phi. \qquad \text{(See page 11.)}$$

The current in a circuit having resistance and inductance is expressed by

$$I = \frac{E}{\sqrt{R^2+(2\pi f L)^2}} \text{ ; from which}$$

$$E = I\sqrt{R^2+(2\pi f L)^2} \text{ ; which is an equation of } \textit{pressures.}$$

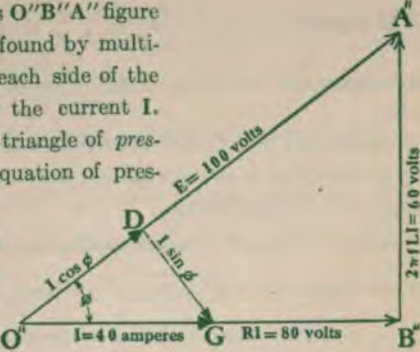

FIG. 40.

Suppose another triangle as **O''B''A''** figure 40 is constructed, to a scale found by multiplying the hypotenuse and each side of the triangle **O A B**, figure 38 by the current **I.** The triangle in figure 40 is a triangle of *pressures*; according to the last equation of pressures. Now suppose the side **O''B''** of the pressure triangle (figure 40) be divided by **R**; that is $\frac{\mathbf{RI}}{\mathbf{R}} = \mathbf{I}$. The numerical value of **I** may be drawn as the vector **O''G.** If the line **GD** is drawn from **G**, the end of the *current* vector, perpendicular to **O''A''** then $\mathbf{O''D} = \mathbf{O''G} \cos\phi = \mathbf{I}\cos\phi$.

In words; **I** cos ϕ denotes the component of the current **I** that is in phase with the applied pressure **E**.

$$\text{But } \mathbf{I} = \frac{\mathbf{E}}{\sqrt{\mathbf{R}^2+(2\pi f\mathbf{L})^2}}\ ; \text{ (in the given case } = \tfrac{100}{2.5} = 40 \text{ amperes.)}$$

$$\text{Then } \mathbf{I}\cos\phi = \frac{\mathbf{E}}{\sqrt{\mathbf{R}^2+(2\pi f\mathbf{L})^2}} \times \cos\phi = \frac{\mathbf{E}}{\mathbf{Z}} \times \cos\phi = \mathbf{E\,Y}\cos\phi,$$

and $\frac{\cos\phi}{\sqrt{\mathbf{R}^2+(2\pi f\mathbf{L})^2}}$, $\frac{1}{\mathbf{Z}}\cos\phi$, and **Y** cos ϕ, each express the *conductance* of the circuit; usually denoted by the letter g. Consult figure 39.

Conductance *may therefore be defined as that factor by which the applied pressure must be multiplied, to give that component of the line current which is in phase with the applied pressure.*

The numerical value of the conductance in the given case is $g = \frac{1}{2.5} \times 0.80003 = 0.32001$.

Again consulting figure 40, it is plain that the side $\mathbf{DG} = \mathbf{O''G}\sin\phi =$ $\mathbf{I}\sin\phi$; but $\mathbf{I}\sin\phi = \frac{\mathbf{E}}{\sqrt{\mathbf{R}^2+(2\pi f\mathbf{L})^2}} \times \sin\phi = \frac{\mathbf{E}}{\mathbf{Z}} \times \sin\phi = \mathbf{EY}\sin\phi$.

I sin ϕ (side **DG**) denotes the component of the line-current **I**, that is *perpendicular* to the applied pressure **E**.

The expressions $\frac{\sin\phi}{\sqrt{\mathbf{R}^2+(2\pi f\mathbf{L})^2}}$, $\frac{1}{\mathbf{Z}}\sin\phi$, and $\mathbf{Y}\sin\phi$, each express the *susceptance* of the circuit; usually denoted by the letter b. See figure 39.

Susceptance *may be defined as that factor by which the applied pressure may be multiplied to give that component of the line current which is perpendicular to the applied pressure.*

The numerical value of the susceptance in the given case is $\frac{1}{2.5} \times 0.59995 = 0.23998$.

From figure 38; $\sqrt{R^2+(2\pi f\mathbf{L})^2} \times \cos\cdot\phi = \mathbf{R}$;

(hypotenuse times the cosine gives the value of the side adjacent.)

$$\text{from which } \cos\phi = \frac{\mathbf{R}}{\sqrt{\mathbf{R}^2+(2\pi f\mathbf{L})^2}} = \frac{\mathbf{R}}{\mathbf{Z}}.$$

But the conductance, $g = \frac{\cos\phi}{\mathbf{Z}}$; therefore,

$$g = \frac{\mathbf{R}}{\mathbf{Z}\times\mathbf{Z}} = \frac{\mathbf{R}}{\mathbf{Z}^2}.$$

(g is always positive; it gives that component of the resultant current that is *in phase* with the applied pressure. No component of a force can be 180° with it.)

Also from figure 38; $\sqrt{\mathbf{R}^2+(2\pi f\mathbf{L})^2} \times \sin\phi = 2\pi f\mathbf{L} = \mathbf{X}$

(hypotenuse times the sine gives the value of the side opposite.)

$$\text{from which; } \sin\phi = \frac{2\pi f\mathbf{L}}{\sqrt{\mathbf{R}^2+(2\pi f\mathbf{L})^2}} = \frac{\mathbf{X}}{\mathbf{Z}}.$$

But the susceptance $b = \frac{\sin\phi}{\sqrt{\mathbf{R}^2+(2\pi f\mathbf{L})^2}} = \frac{\sin\phi}{\mathbf{Z}}$; therefore

$$b = \frac{\mathbf{X}}{\mathbf{Z}\times\mathbf{Z}} = \frac{\mathbf{X}}{\mathbf{Z}^2}.$$

(If **X** is *negative* then the value of b is negative; sometimes the current *leads* the pressure, sometimes it lags the pressure; therefore, unlike g, b may be negative as well as positive.)

The algebraical sign of $\cos\phi$, from the expression $\frac{\mathbf{R}}{\sqrt{\mathbf{R}^2+(2\pi f\mathbf{L})^2}} = \frac{\mathbf{R}}{\mathbf{Z}}$ is always + since **R** *is always positive*.

The sign of $\sin\phi$ is + when the capacity of the circuit is infinite; that is when $\frac{1}{2\pi f\mathbf{C}} = 0$.

If the capacity of the line is *less* than ∞ then

$$\sin \phi = \frac{2\pi f \mathbf{L} - \frac{1}{2\pi f \mathbf{C}}}{\sqrt{\mathbf{R}^2 + (2\pi f \mathbf{L})^2}} = \frac{\mathbf{X}}{\mathbf{Z}}.$$

In this case if $\frac{1}{2\pi f \mathbf{C}}$ is *greater* than $2\pi f \mathbf{L}$ then **X** is negative and $\sin \phi = -\frac{\mathbf{X}}{\mathbf{Z}}$; the susceptance b is then *negative* and must be so treated when combining it with other susceptances.

TWO IMPEDANCES IN PARALLEL.

Suppose two impedances (such as two coils), are connected together in parallel as indicated in figure 41; let one have a resistance denoted by $\mathbf{R}_,$ and a coefficient of inductance denoted by $\mathbf{L}_,$; while the other has a resistance denoted by $\mathbf{R}_{,,}$ and a coefficient of inductance denoted by $\mathbf{L}_{,,}$.

Let an alternating-pressure denoted by **E** be applied to the terminals **AB** of the *arrangement* as indicated. The frequency of the pressure will be denoted by f.

The pressure E is common to both branches of the arrangement, and produces a current in the upper branch denoted by $\mathbf{I}_,$ and a current in the lower branch denoted by $\mathbf{I}_{,,}$. The current in the main line will be denoted by **I.** $\mathbf{I}_,$ and $\mathbf{I}_{,,}$ are not necessarily equal to each other and not necessarily in the same phase relation with the applied pressure **E.**

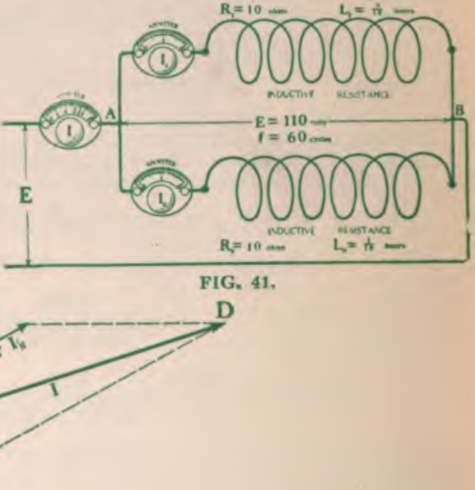

FIG. 41.

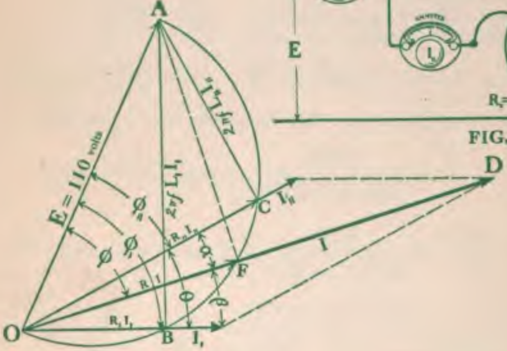

FIG. 42.

The current in the upper branch may be expressed by:

$\mathbf{I}_, = \dfrac{\mathbf{E}}{\sqrt{\mathbf{R}_,^2 + (2\pi f \mathbf{L}_,)^2}}$ The angle of *lag* of this current, from **E**, is found from:

$$\tan\phi_, = \frac{2\pi f \mathbf{L}_, \mathbf{I}_,}{\mathbf{R}_, \mathbf{I}_,} \;;\; \text{or } \frac{2\pi f \mathbf{L}_,}{\mathbf{R}_,}. \quad \text{(Canceling } \mathbf{I}_,\text{.)}$$

The current in the lower branch may be expressed by:

$$\mathbf{I}_{,,} = \frac{\mathbf{E}}{\sqrt{\mathbf{R}_{,,}^{2}+(2\pi f\,\mathbf{L}_{,,})^{2}}}\ ;$$

The angle of lag of this current from **E** may be found from:

$$\tan\phi_{,,} = \frac{2\pi f\,\mathbf{L}_{,,}\,\mathbf{I}_{,,}}{\mathbf{R}_{,,}\mathbf{I}_{,,}}\ ;\ \text{or}\ \frac{2\pi f\,\mathbf{L}_{,,}}{\mathbf{R}_{,,}}.\quad (\text{Canceling } \mathbf{I}_{,,}.)$$

Consulting figure 42, page 142, $\phi_{,}$ is the angle of lag of the current vector $\mathbf{I}_{,}$ (along **OB**) behind the applied pressure vector **E** (**OA**). The vector **AB** represents the reactive pressure $2\pi f\mathbf{L}_{,}\mathbf{I}_{,}$ which is at right angles with $\mathbf{R}_{,}\mathbf{I}_{,}$, and the triangle **OBA** is therefore inscribed in a semicircle whose diameter is **E** (**OA**). **E** is the hypotenuse of the right triangle **OBA.**

The tangent of $\phi_{,}$ is the side opposite $\phi_{,}$, divided by the side adjacent to $\phi_{,}$; or $\tan\phi_{,} = \dfrac{2\pi f\mathbf{L}_{,}\,\mathbf{I}_{,}}{\mathbf{R}_{,}\mathbf{I}_{,}}$.

The numerical value of $\mathbf{R}_{,}\mathbf{I}_{,}$ divided by the numerical value of $\mathbf{R}_{,}$ will give the numerical value of the current $\mathbf{I}_{,}$; which *might be greater* (numerically) than $\mathbf{R}_{,}\mathbf{I}_{,}$; if $\mathbf{R}_{,}$ is *less than unity*; that is less than one ohm. In the figure the current vector $\mathbf{I}_{,}$ is shown longer than the vector $\mathbf{R}_{,}\mathbf{I}_{,}$. This is also true of other current vectors in the same diagram and is adopted for convenience and clearness.

In the same diagram, $\phi_{,,}$ is the angle of lag of the current vector $\mathbf{I}_{,,}$ (along **OC**) behind the applied pressure **E** (**OA**). The vector **AC** represents the reactive pressure, $2\pi f\mathbf{L}_{,,}\mathbf{I}_{,,}$; is at right angles with $\mathbf{R}_{,,}\mathbf{I}_{,,}$, and the right triangle **OCA** is inscribed in the semicircle whose diameter **E** is the hypotenuse **OA** of the right triangle **OCA.**

The tangent of $\phi_{,,}$ is the side opposite $\phi_{,,}$ ($\mathbf{AC} = 2\pi f\mathbf{L}_{,,}\mathbf{I}_{,,}$) divided by the side adjacent to $\phi_{,,}$; ($\mathbf{OC} = \mathbf{R}_{,,}\,\mathbf{I}_{,,}$).

$$\text{Or } \tan\phi_{,,} = \frac{2\pi f\,\mathbf{L}_{,,}\,\mathbf{I}_{,,}}{\mathbf{R}_{,,}\,\mathbf{I}_{,,}}\ ;\ \text{or}\ \frac{2\pi f\mathbf{L}_{,,}}{\mathbf{R}_{,,}}.\qquad (\text{Canceling } \mathbf{I}_{,,}.)$$

The current vector $\mathbf{I}_{,,}$ is obtained by dividing $\mathbf{R}_{,,}\mathbf{I}_{,,}$ by $\mathbf{R}_{,,}$.

If the two coils were similar; $\mathbf{R}_{,} = \mathbf{R}_{,,}$ and $\mathbf{L}_{,} = \mathbf{L}_{,,}$, then $\phi_{,}$ would be equal to $\phi_{,,}$, $\mathbf{I}_{,}$ would be equal to $\mathbf{I}_{,,}$ and $\mathbf{I}_{,}+\mathbf{I}_{,,} = \mathbf{I}$. In this case θ becomes 0.

It is evident further, that $\phi_{,,}$ might be larger than $\phi_{,}$; in which case the angle θ would be the *difference* between $\phi_{,,}$ and $\phi_{,}$.

The vector resultant, (**I**) of $\mathbf{I}_{,}$ and $\mathbf{I}_{,,}$ is denoted by the line **OD.**

The resultant current (**I**) in the main line is expressed by:

$\mathbf{I} = \sqrt{\mathbf{I}_{,}^{2}+\mathbf{I}_{,,}^{2}+2\mathbf{I}_{,}\mathbf{I}_{,,}\cos\theta}$ in which θ is the angle between $\mathbf{I}_{,}$ and $\mathbf{I}_{,,}$; see figure 42. The angle θ is equal to $\phi_{,} - \phi_{,,}$. (θ is also equal to angle α+angle β.)

The value, in degrees, of the angle $(\phi_{,,}+\alpha)$ of lag of **I** from **E** may be found from:

$$\cos\alpha = -\frac{I_{,}^2 - I_{,,}^2 - I^2}{2\,I \times I_{,,}}. \qquad \text{(See page 29.)}$$

From the evaluation of the above, after substituting the values of the various currents, the angle corresponding with the numerical value of the cosine thus found, may be obtained from tables of natural trigonometrical functions. Then the lag of **I** behind **E** is $\phi = \phi_{,,} + \alpha$.

It is evident that by the same method, the value, in degrees, of the angle β may be found, from which;

$$\phi = \phi_{,} - \beta.$$

If the letter **Z** is used to denote impedance, and **Y** to denote admittance, then:

$$I_{,} = \frac{E}{Z_{,}} = EY_{,};$$

$$I_{,,} = \frac{E}{Z_{,,}} = EY_{,,},$$

$$\text{and}\quad I = \frac{E}{Z} = EY.$$

Substituting these values in the equation for **I**, gives:

$$I = \sqrt{E^2Y_{,}^2 + E^2Y_{,,}^2 + 2EY_{,}EY_{,,}\cos\theta} = E\sqrt{Y_{,}^2 + Y_{,,}^2 + 2Y_{,}Y_{,,}\cos\theta}.$$

Since it is true that:

$$I = \frac{E}{Z} = EY,$$

$$\text{then;}\quad EY = E\sqrt{Y_{,}^2 + Y_{,,}^2 + 2Y_{,}Y_{,,}\cos\theta}.$$

$$\text{or}\quad Y = \sqrt{Y_{,}^2 + Y_{,,}^2 + 2Y_{,}Y_{,,}\cos\theta}.$$

This proves that the *joint admittance* or the *equivalent admittance* of two impedances connected together in parallel, is the *vectorial sum* of the admittances of the individual impedances, and that **the angle between admittances is the same as the angle between corresponding currents.** Consult page 143. This fact is of very great importance, as it forms a basis for the plotting and the evaluation of component currents and pressures.

The foregoing analysis shows two methods of finding the *resultant* or main-line current in a divided circuit consisting of resistances and inductances *in parallel*.

1. The value of each branch current may be found.
The angle between the two branch currents may be found.
The resultant of the two currents may be found.

The angle between the resultant current and the applied pressure may be found; all according to laws applying to the resultant of two forces, (as in "Mechanics") involving trigonometrical relations.

2. The numerical value of each impedance may be found.

The reciprocal of each impedance gives the admittance of each branch.

The angle between the admittances is found as with the currents.

The *resultant* of the admittances is found (vectorially).

The value of the resultant admittance multiplied by the applied pressure gives the numerical value of the resultant or main line current.

The angle between the EQUIVALENT ADMITTANCE and the applied pressure is the same as the angle between the resultant current and the applied pressure.

Referring again to figure 42, page 142 it will be noted that the resultant current vector intersects the semicircle at point **F**. The vector **OF** may be denoted by **RI**. The numerical value of **RI** (or **OF**) may be found from: $\mathbf{RI} = \mathbf{E} \cos \phi$.

It is evident that $\mathbf{R} = \frac{\mathbf{E}}{\mathbf{I}} \cos \phi$. **R** may be termed the *A.C.* equivalent resistance* of the two branch circuits or of the two impedances.

The vector **AF**, denotes the **A. C.** equivalent reactive pressure, and

$$2\pi f\mathbf{LI} = \mathbf{E} \sin \phi;$$

from which

$$2\pi f\mathbf{L} = \frac{\mathbf{E}}{\mathbf{I}} \sin \phi.$$

$2\pi f\mathbf{L}$ denotes the *equivalent* reactance of the two branch circuits or of the two impedances which are connected together in parallel.

This means that a *single coil* having an inductance denoted by **L** and a resistance denoted by **R**, could replace the two coils mentioned, allowing the resultant current **I** to obtain.

In solving alternating-current problems involving the various connections of resistances, inductances and capacities, and the corresponding relations of the various "vector" quantities, it will be found advantageous to draw figures such as are indicated in figures 41 and 42 page 142, affixing to the proper portions of each diagram the known values of the various quantities involved. Such diagrams serve to effect a comprehensive insight that can be realized in no other way. To illustrate the fundamental relationship of the various quantities, it is not necessary to draw such preliminary diagrams to exact scale. Too much dependence should not be placed upon results which are measured on scale diagrams. Under certain conditions diagrams cannot furnish accurate numerical results as will be indicated later.

***Alternating-Current.**

PARALLEL ARRANGEMENTS.

EXAMPLE 38: Suppose a non-inductive resistance of 65 ohms is connected in parallel with an inductive resistance of 5 ohms and $\frac{2}{10}$ henry, and a sine-pressure of 110 volts (effective) at 60 cycles is applied to the terminals of the arrangement; find the current in the main line and its phase angle referred to the applied pressure.

The current in the inductive branch may be denoted by $\mathbf{I}_{,}$ and the current in the non-inductive branch by $\mathbf{I}_{,,}$.

GIVEN DATA.

$E=110$ volts (effective value)
$f=60$ cycles.
$\mathbf{R}_{,}=5$ ohms.
$\mathbf{L}_{,}=\frac{2}{10}$ henry.
$\mathbf{R}_{,,}=65$ ohms.

Then:

$$\mathbf{I}_{,}=\frac{E}{\sqrt{R_{,}^2+(2\pi f L_{,})^2}}$$

$$=\frac{110}{\sqrt{25+5685.16}}$$

$$=\tfrac{110}{75.5}=\mathbf{1.45}\text{ AMPERES,}$$

and

$$\mathbf{I}_{,,}=\frac{E}{R_{,,}}=\tfrac{110}{65}=\mathbf{1.69}\text{ AMPERES.}$$

Computed Data.

$\mathbf{I}_{,}=1.45$ amperes $\phi_{,}=86°12'$
$\mathbf{I}_{,,}=1.69$ amperes $\phi_{,,}=0°$
$2\pi f=377$ (See page 208.)
$2\pi f L_{,}=75.4$.

The current $\mathbf{I}_{,}$ lags behind the applied pressure at an angle which may be designated by $\phi_{,}$, and

$$\tan\ \phi_{,}=\frac{2\pi f L_{,}}{\mathbf{R}_{,}}=\tfrac{75.4}{5}=15.08\text{: from which}$$

$$\phi_{,}=86°\ 12'.\ \cos\ 86°\ 12'=0.06627.$$

The current $\mathbf{I}_{,,}$ is in phase with the applied pressure **E**. The angle between $\mathbf{I}_{,}$ and $\mathbf{I}_{,,}$ is therefore 86° 12′.

The resultant current in the main line is expressed by:

$$\mathbf{I}=\sqrt{\mathbf{I}_{,}^2+\mathbf{I}_{,,}^2+2\,\mathbf{I}_{,}\,\mathbf{I}_{,,}\cos 86°\ 12'}.$$

$$=\sqrt{1.45^2+1.69^2+2\times 1.45\times 1.69\times 0.06627}.$$

$$=\mathbf{2.29\ amperes.}\quad\textbf{Answer.}$$

The angle of lag of the resultant line current **I** behind the applied pressure **E** may be found from the application of the trigonometrical principle stated on page 29.

Letting ϕ denote the angle between the resultant current and the applied pressure then:

$$\cos\phi = -\frac{\mathbf{I}_{\prime}^{2}-\mathbf{I}^{2}-\mathbf{I}_{\prime\prime}^{2}}{2\mathbf{I}\mathbf{I}_{\prime\prime}}$$

$$= -\frac{2.0925 - 5.2734 - 2.8561}{2 \times 2.29 \times 1.69} = 1.0218 \text{ from which}$$

$$\phi = 45^\circ\ 47'.$$

PROBLEM 38: If a non-inductive resistance of 100 ohms is connected in parallel with an inductive resistance of 10 ohms and 1 henry, and an A.C. pressure of 151 volts at 60 cycles is applied to the terminals of the arrangement; find the maximum value of the current in the main line, and its phase angle, in degrees, referred to the applied pressure.

$\mathbf{I_{max} = 0.41}$ ampere. $\phi = 73^\circ\ 44'$ **Answer.**

Problem 38a: Suppose a non-inductive resistance of 50 ohms is connected in parallel with an inductive resistance of $2\frac{1}{2}$ ohms and $\frac{1}{10}$ henry, and an alternating-pressure of 110 volts at 60 cycles is applied to the terminals of the arrangement; find the current in the main line, and its phase angle, in degrees, referred to the applied pressure.

Problem 38b: Suppose a non-inductive resistance of 65 ohms, is connected in parallel with an inductive resistance of 5 ohms and $\frac{2}{10}$ henry, and a sine-pressure of 110 volts, (effective) at 25 cycles is applied to the terminals of the arrangement; find the current in the main line and its phase angle with reference to the applied pressure.

Problem 38c: Plot the results of EXAMPLE 38 in the form of a vector diagram, showing all pressure and current vectors. Use a scale of 1 inch = 10 volts and 1 inch = 2 amperes.

PARALLEL SERIES ARRANGEMENTS OF INDUCTIVE AND NON-INDUCTIVE RESISTANCES.

EXAMPLE 39: Construct a vector diagram, to scale, showing the pressure and current relations in an arrangement as indicated by figure 43, having values as there stated.

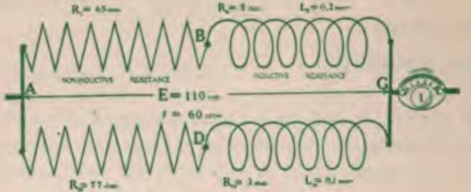

FIG. 43.

The current in the branch **ABG** may be denoted by $\mathbf{I_B}$, the current in the branch **ADG** by $\mathbf{I_D}$ and the resultant of these two currents (the current in the main line), by **I**.

Since **I** is the vectorial resultant of $\mathbf{I_B}$ and $\mathbf{I_D}$, it will be necessary to find the numerical values of $\mathbf{I_B}$, $\mathbf{I_D}$, and the angle between them. This angle may be denoted by θ.

GIVEN DATA.

$\mathbf{E}=110$ volts.
$f=60$ cycles.
$\mathbf{R}_{,,}=5$ ohms.
$\mathbf{L}_{,,}=\frac{2}{10}$ henry.
$\mathbf{R}_{,}=65$ ohms.
$\mathbf{R}_{,,,}=3$ ohms.
$\mathbf{L_{iv}}=\frac{1}{10}$ henry.
$\mathbf{R_{iv}}=77$ ohms.

Computed Data.

$2\pi f=377.$

The numerical value of $\mathbf{I_B}$ may be found from:

$$\mathbf{I_B}=\frac{110}{\sqrt{(65+5)^2+(377\times\frac{2}{10})^2}}=\frac{110}{\sqrt{4900+5685.16}}$$

$$=\frac{110}{102.8}=\mathbf{1.07\ Ampere.}$$

The angle of lag $\phi_\mathbf{B}$ of this current behind the applied pressure **E** is found from:

$$\tan\phi_\mathbf{B}=\frac{2\pi f\mathrm{L}_{,}}{\mathrm{R}_{,}+\mathrm{R}_{,,}}=\frac{75.4}{70}=1.07714.$$

$$\phi_\mathbf{B}=47°\ 8'.$$

The pressure component between **A** and **B** is

$$\mathbf{R}_{,,}\ \mathbf{I}_{\mathbf{B}} = 65 \times 1.07 = \mathbf{69.55\ Volts.}$$

The pressure component between **B** and **G** is

$$\mathbf{I}_{\mathbf{B}}\sqrt{\mathbf{R}_{,}^{2}+(2\pi f\mathbf{L}_{,})^{2}} = 1.07\sqrt{25+\overline{75.4}^{2}} = 1.07 \times 75.5 = 80.78\ \mathbf{Volts.}$$

The numerical value of $\mathbf{I}_{\mathbf{D}}$ may be found from:

$$\mathbf{I}_{\mathbf{D}} = \frac{110}{\sqrt{(77+3)^{2}+(377\times\frac{1}{10})^{2}}} = \frac{110}{\sqrt{6400+1421.29}}$$

$$= \frac{110}{88.4} = \mathbf{1.24\ Amperes.}$$

The angle of lag $\phi_{\mathbf{D}}$ of $\mathbf{I}_{\mathbf{D}}$ behind the applied pressure **E** is found from:

$$\tan\phi_{\mathbf{D}} = \frac{2\pi f\mathbf{L}_{,,,}}{\mathbf{R}_{,,,}+\mathbf{R}_{\mathbf{iv}}} = \frac{37.7}{80} = 0.47125.$$

$$\phi_{\mathbf{D}} = 25^{\circ}\ 14'.$$

The pressure component between **A** and **D** is

$$\mathbf{R}_{,,}\mathbf{I}_{\mathbf{D}} = 77\times1.24 = \mathbf{95.48\ Volts.}$$

The pressure component between **D** and **C** is

$$\mathbf{I}_{\mathbf{D}}\sqrt{\mathbf{R}_{,,,}^{2}+(2\pi f\mathbf{L}_{,,,,})^{2}} = 1.24\sqrt{9+\overline{37.7}^{2}}$$

$$=1.24\sqrt{1430.29}$$

$$=1.24 \times 37.8$$

$$=\mathbf{46.87\ Volts.}$$

The angle θ, between $\mathbf{I}_{\mathbf{B}}$ and $\mathbf{I}_{\mathbf{D}}$ is $=(47^{\circ}\,8'-25^{\circ}\,14')=21^{\circ}\,54'$.

$$\cos 21^{\circ}\,54' = 0.92784.$$

The resultant or main line current is found from:

$$\mathbf{I} = \sqrt{\mathbf{I}_{\mathbf{B}}^{2}+\mathbf{I}_{\mathbf{D}}^{2}+2\mathbf{I}_{\mathbf{B}}\mathbf{I}_{\mathbf{D}}\cos\theta}.$$

$$= \sqrt{\overline{1.07}^{2}+\overline{1.24}^{2}+2\times1.07\times1.24\times0.92784}.$$

$$=\mathbf{2.26\ Amperes.}$$

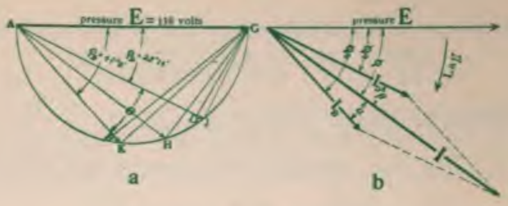

FIG. 44.

The results are shown plotted to scale in figure 44 (**a**) and (**b**); (**a**) showing the pressure vectors and (**b**) an enlarged diagram showing the current relations.

The angle of lag ϕ of the current **I** behind the applied pressure **E** may be found by first finding the value of β from;

$$\cos\beta = -\frac{\mathbf{I_B}^2-\mathbf{I_D}^2-\mathbf{I}^2}{2\,\mathbf{I_D}\,\mathbf{I}} = 0.98135;\ \text{from which}\ \beta = 11^\circ\,5'$$

Therefore, $\phi=(\phi_{\mathrm{D}}+\beta)=36^\circ\,19'$.

The angle α in figure 44 **b** could have been found; from which

$\phi=\phi_{\mathrm{B}}-\alpha$; which would serve as a check.

For interpretation of different types of arrow heads employed, consult page 183.

PROBLEM 39: Given non-inductive resistances and inductive resistances arranged as indicated in figure 43, page 48, *but having the following values:* $\mathbf{R}_{,}=5$ *ohms,* $\mathbf{L}_{,}=\frac{2}{10}$ *henry,* $\mathbf{R}_{,,}=45$ *ohms,* $\mathbf{R}_{,,,}=3$ *ohms,* $\mathbf{L}_{,,,}=\frac{1}{100}$ *henry,* $\mathbf{R}_{\mathrm{iv}}=147$ *ohms; and the applied pressure* $\mathbf{E}=110$ *volts at 60 cycles. Compute all values as in* EXAMPLE *39. Check all values with a diagram drawn to scale.*

$\mathbf{I_B}=1.21$ AMPERE, $\mathbf{I_D}=0.73$ AMPERE.
$\mathbf{I}=1.74$ AMPERE, $\Theta=55^\circ\,1'$.
$\mathbf{R}_{,,}\mathbf{I_B}=54.67$ VOLTS, $\mathbf{I_B}\sqrt{\mathbf{R}_{,}^2+(2\pi f\,\mathbf{L}_{,})^2}=91.85$ VOLTS.
$\mathbf{R}_{,,}\mathbf{I_D}=107.25$ VOLTS, $\mathbf{I_D}\sqrt{\mathbf{R}_{,,,}^2+(2\pi f\mathbf{L}_{,,,})^2}=3.51$ VOLTS.

Phase of **I** behind $\mathbf{E}=(34^\circ\,37'+1^\circ\,26')=36^\circ\,3'$.

Problem 39a: Compute the values and construct a vector diagram to scale, letting 1 inch denote 10 volts, given the arrangement as in EXAMPLE 39, using all values there given except the frequency. Change the frequency to 25∼. The separate current diagram may be drawn using a scale of 1 inch to denote $\frac{1}{2}$ ampere.

PROBLEM 39b: Compute the values and construct a vector diagram to scale, showing the relations of the various pressures and currents in an arrangement as shown in figure 43, *page* 148, *with the numerical values as given in PROBLEM 39, except the frequency; change this to* 25 *cycles.*

$\mathbf{I_B} = \mathbf{1.433}$ **ampere.** $\phi_B = 24° 10'$ $\cos\theta = 0.97417$.

$\mathbf{I_D} = \mathbf{1.349}$ **ampere.** $\phi_D = 11° 7'$

$\mathbf{I} = \mathbf{2.76}$ **amperes.** $\theta = 13° 3'$ **Answer.**

FUNDAMENTAL CONSIDERATIONS OF CAPACITY.

Capacity of a Condenser. The electric capacity of a conductor is defined as the ratio of its electrical charge to its potential, when so situated as to be uninfluenced by its environments; or in other words when *all* other conductors near it are at zero potential. Potential as here referred to, means the electrical pressure in *absolute static units*, above zero.

Capacity may then be expressed by symbols as:—

$C = \frac{Q}{V}$; in which equation C denotes the capacity, expressed in practical units; Q denotes the quantity of electricity, in coulombs, and V denotes the difference in potential in volts.

From the above equation it is evident that $Q = CV$ and $V = \frac{Q}{C}$, and further that the capacity of a condenser is a *constant*.

Stated in words, the **numerical value of** the electrical **capacity** of a conductor, in practical units, **is the number of coulombs of electricity necessary to raise** its potential, or **its electrical pressure,** from zero to **one volt.**

The practical unit of capacity is called the **farad,** and the letter C denotes capacity in farads when used in equations in this book; unless otherwise stated. In ordinary engineering practice, the farad is too large a unit for convenience, therefore a sub-multiple of it is used, called the microfarad. One microfarad $= \frac{1}{1000000} = .000001$ of one farad.

A condenser may be said to have *unit* capacity, in any system of units, if unit quantity of electricity supplied to it increases the pressure between its terminals one unit.

The element of time does not enter into the consideration; the electrical quantity (coulombs) may be supplied during $\frac{1}{100}$ of a second, or during 100 years.

As defined above it might appear that there is no limit to the amount of electricity (coulombs) that may be put onto an insulated conductor. The limitation is, however, when the pressure, indicated by V, becomes great enough to puncture the dielectric surrounding the conductor, causing a "spark discharge"; during which phenomenon, the *charge* on the conductor nearly or entirely disappears.

The capacity of a continuous conductor is infinite. To illustrate this statement: suppose the *ends* of an insulated conductor be connected with the terminals of a battery: a certain current results; in other words a certain *quantity* of electricity in coulombs, passes a given point on the connecting wire every second, onto the conductor, but the *potential* of the conductor in this case does not continue to *increase*.

This consideration has an important bearing in relation to the general equation for the current in any circuit, when an alternating-pressure is applied to the circuit; namely to:—

$$I = \frac{E}{\sqrt{R^2+(2\pi f L - \frac{1}{2\pi f C})^2}};$$

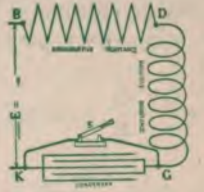

FIG. 47.

which is applicable when there is a resistance, an inductance and a capacity connected together *in series*, as indicated in figure 47. Suppose that one positive terminal **C** or plate of the condenser comes into contact with a negative plate, as by closing the switch **S**. The capacity of the circuit now becomes *infinite*; or in the above equation, the *denominator* of the last term under the radical, containing **C**, becomes infinite, and the whole term becomes *zero*. The equation then appears as;

$$I = \frac{E}{\sqrt{R^2+(2\pi f L)^2}};$$ which applies to a circuit containing only resistance and inductance. The condenser is short circuited in this case; which amounts to removing the condenser from the circuit.

Capacity of a Condenser in Terms of its Dimensions. The capacity in absolute units, of a metal sphere, situated in air, and removed from the influence of external electric fields, is numerically equal to its radius; the length of the radius being expressed in centimeters. This results from the fact that the potential of the sphere is:

$$V = \frac{Q}{r}, \text{ and}$$

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{r}} = r,$$ where **r** denotes the radius of the sphere in *centimeters*.

Let figure 48 denote, in cross section, two isolated, insulated, and concentric metallic spheres, separated by an air space, one insulated from the earth by being suspended by a silk thread, S; while the outer sphere is in good metallic connection with the earth; the radii of the two spheres being r and $r_,$ and one sphere having a certain amount of positive charge while the other has an equal negative charge induced upon it, as indicated.

The potential at the center **O** due to the outer charge is expressed by $-\frac{Q}{r_,}$, while the potential at the center **O**, due to the inner charge is expressed by $\frac{Q}{r}$. The *resultant potential* at the center is the sum of the two potentials or;

$$V_\circ = \frac{Q}{r} - \frac{Q}{r_,}, \text{ from which is obtained:}$$

$$V_\circ = Q\left(\frac{1}{r} - \frac{1}{r_,}\right) = Q\left(\frac{r_, - r}{rr_,}\right) \quad \text{and}$$

$$\frac{Q}{V_\circ} = \frac{rr_,}{r_, - r}.$$

Let $r_, - r = t$: whence $r_, = t + r$;

$$\text{then } C = \frac{Q}{V} = \frac{r_,(t+r)}{t}; \text{ or very nearly } = \frac{r_, r.}{t}$$

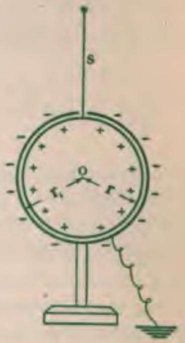

FIG. 48.

The distance, t, between the spheres is very small compared with the radius r, so that r and $r_,$, are very nearly equal, and $C = \frac{r^2}{t}$ is therefore very nearly correct.

The following relation must then hold.

$$C = \frac{4\pi r^2}{4\pi t} \text{ (multiply both numerator and denominator by } 4\pi)$$

But $4\pi r^2$ expresses the area of the inner sphere in terms of its radius. Denoting this area by **A** then

$$C = \frac{A}{4\pi t}, \textit{ electro-static units.}$$

If the size of the spheres is increased until **r** (and $r_,$) becomes infinite, while the distance between the two spheres is not changed, the surfaces of the spheres become planes and the last expression will denote the capacity of two *parallel plates*, separated a distance t; the area of one plate being **A**.

If there is a medium other than air between the two plates, whose specific inductive capacity, instead of being **1** as for air, has a different value, which may be denoted by k, then the capacity may be expressed by

$$C = \frac{k}{4\pi} \times \frac{A}{t} \text{ \textit{electro-static units.}}$$

Since the practical unit, the farad, is equivalent to 9×10^{11} electro-static units of capacity, the general expression for the capacity of any condenser becomes:—

$$\mathbf{C} = \frac{k\ \ A}{4\pi t \times 9 \times 10^{11}}\text{; IN FARADS.}$$

$$= \frac{k\ \ A}{1.13097 \times 10^{13} t}\text{ ; IN FARADS.}$$

$$= .0000000000000884\,\frac{k\,A}{t}\text{ ; IN FARADS.}$$

Since a microfarad $= \frac{1}{1,000,000}$ of a farad the above becomes

$$= .0000000884\,\frac{k\,A}{t}\text{; IN MICROFARADS;}$$ when the area **A** is expressed in *square centimeters* and t is expressed in *centimeters*. The capacity may be expressed in terms of inches by introducing the relation of the English to the metric system of units as follows:—

$$\mathbf{C} = .0000000884\,\frac{\overline{2.5399}^2}{2.5399}\,\frac{k\,A}{t}\text{ in \textit{microfarads.}}$$

$$= .000000224\,\frac{A}{t}\,k\text{; in \textit{microfarads}; in which } \mathbf{A} \text{ denotes}$$

square inches, and t denotes distance between the plates, in inches.

Since the dielectric usually employed between the plates, in making condensers, is very thin, t is conveniently expressed in *mils* or *thousandths* of an inch; if n plates are used, the common working equation for capacity is obtained by dividing the last equation by 1000, and multiplying it by n, giving;

$$\mathbf{C} = .000224\,\frac{An}{t}\,k\text{; IN MICROFARADS.}$$

In which the area A, of one plate or of one side of the dielectric, is expressed in *square inches* and the thickness t, of the *dielectric* is expressed in *mils*; n denotes the number of insulating plates or the *number oj dielectric sheets* employed. Two plates of conductors will necessitate one sheet of dielectric: three conductor plates, two sheets of dielectric, etc.

The application of the last equation in practice will be explained in the next example.

CAPACITY OF CONDENSERS IN TERMS OF THEIR DIMENSIONS.

EXAMPLE 41: A condenser consists of 263 plates of tin-foil, each plate 10 inches × 12 inches; separated by crown glass plates $\frac{15}{100}$ of an inch thick. Find the capacity of the condenser, in farads.

$C = .000225 \frac{A\,n}{t}\,k$ microfarads.

$= .000225\ \frac{120 \times 262}{150}\ 6.96$

$= .328$ microfarads.

$=$ **.000000328 FARAD. Answer.**

DATA.

$A = 10'' \times 12'' = 120$ sq.″

$k = 6.96$ for crown glass.

$t = \frac{15}{100} = 150$ mils.

$n = 262$ sheets of dielectric.

One farad is equal to 1,000,000 microfarads.

PROBLEM 41: Find the number of sheets of paraffined paper .015 of an inch thick, that will be required to make a condenser using tin foil plates 6 inches long and 20 inches wide, that shall have a capacity of 3.28 microfarads. **907 plates. Answer.**

PROBLEM 41a: Find the capacity, expressed in microfarads, of a condenser made up of 12 *brass or copper plates,* 6×10 *inches, separated, in air, a distance of ½ inch.* **.00324 mf. Answer.**

Problem 41b: Suppose a condenser is made of metal plates 6 inches wide and that overlap each other 10 inches, with dielectric of crown glass ⅛-inch thick, and with one set of plates as **B** figure 49 arranged to be pulled out away from the **A** set.

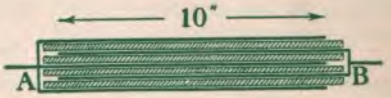

FIG. 49.

Find the decrease in the capacity of the arrangement if the **B** set is moved from its initial position a distance of 1 inch.

Problem 41a and Problem 41b have a direct application to wireless telegraph condensers, for the sending circuit.

Attention is directed to the fact that the dielectric strength of an insulator and the specific inductive capacity of the same, are very different. The dielectric strength means the resistance offered by the material of the insulator to puncture; when the material is placed between needle points and very high pressures applied.

Specific inductive capacity means the increase (or decrease in case of vacuum and alcohol) in capacity of two or more metallic plates, when insulating material other than air is placed between them.

As indicated in the table of specific inductive capacities, below, if flint glass is placed between two metallic plates their capacity will be 9.9 times as much as when the glass is removed and air is between the plates; the distance between the plates being unchanged.

Those desiring to construct condensers, that shall have definite capacities, according to the equation on page 155, will find the following table of specific inductive capacities of considerable value.

TABLE OF SPECIFIC INDUCTIVE CAPACITY OF DIELECTRICS.

Air	1.
Hydrogen	0.9997
Vacuum	0.9994
Ethyl Alcohol	32.65
Castor Oil	4.7
Colza Oil	3.1
Olive Oil	3.1
Crude Petroleum	2.07
Vaseline	2.17
Flint glass having density of 4.5	9.9
Crown glass having density of 2.485	6.96
Plate glass	3.31 to 8.4
Mica	6. to 7
Paraffin	2.2
Paraffined paper	2.2
Rosin	2.5
Rock Salt	18.
Shellac	3.2
Bees Wax	4.1
Ebonite	2.8

While the resistance of a condenser is commonly spoken of as *infinite*, this is not strickly true. The resistance of even a good dielectric such as is used in constructing condensers, must have some finite value. The resistance however is so great it is expressed in megohms.

PHASE RELATION OF CHARGING CURRENT OF A CONDENSER.

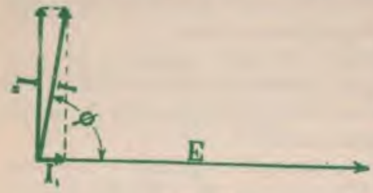

FIG. 50.

The fact that the resistance of the dielectric of a condenser is less than an infinite value really establishes the phase relation of the charging current as *less* than 90° in advance of the pressure applied to the terminals of a condenser. The actual condition may be illustrated by figure 50 in which the vector **E** denotes the applied pressure **E**; while **I** denotes the current **I** in the wires leading to the condenser. The current **I** may be considered as made up of two components at right angles with one another; one of which, denoted by $\mathbf{I}_{,}$, is in phase with the applied pressure **E**; while the other denoted by $\mathbf{I}_{,,}$ is perpendicular to **E**.

Then $I^2 = I_{,}^2 + I_{,,}^2$. If $\mathbf{I}_{,}$ is in phase with **E**, then $\mathbf{I}_{,} = \frac{\mathbf{E}}{\mathbf{R}}$; in which **R** denotes the true resistance of the condenser in *megohms*. Obviously with $\mathbf{I}_{,}^2 = \frac{\mathbf{E}^2}{\mathbf{R}^2}$. Further; $\mathbf{I}_{,,} = 2\pi f\mathbf{CE}$, and $\mathbf{I}_{,,}^2 = (2\pi f\mathbf{CE})^2$. Therefore;

$$\mathbf{I}^2 = \frac{\mathbf{E}^2}{\mathbf{R}^2} + (2\pi f\mathbf{CE})^2.$$

The current may be expressed as;

$$\mathbf{I} = \sqrt{\frac{\mathbf{E}^2}{\mathbf{R}^2} + (2\pi f\mathbf{CE})^2}$$

The applied pressure, denoted by **E** is usually not of great magnitude; say about 110 volts, so that $\mathbf{E}^2$ is 12100. The resistance of the condenser is however very great; usually expressed in megohms; *millions* of ohms. Even though the resistance of a condenser is only one megohm, 1,000,000 ohms, this number squared is *very* large compared with 12100. In general it is obvious that $\frac{\mathbf{E}^2}{\mathbf{R}^2}$ is *very small*, and may be neglected in relation with $(2\pi f\mathbf{CE})^2$, so that the current may be expressed as;

$$\mathbf{I} = \sqrt{(2\pi f\mathbf{CE})^2} = 2\pi f\mathbf{CE};$$ for all practical purposes.

The component $\mathbf{I}_{\prime}$ which is in phase with the applied pressure, represents the energy component of the total current; the power in heating the condenser being expressed by $\mathbf{EI}_{\prime}$; or by $\mathbf{RI}_{\prime}^{2}$. The component $\mathbf{I}_{\prime\prime}=2\pi f\mathbf{CE}$ represent a *wattless* component; being (vectorially) perpendicular to the applied pressure **E**.

The greater the value of the energy component of the current the greater the power loss due to heating the dielectric of the condenser.

If ϕ denotes the angle (of lead) between the resultant current **I** and the applied pressure **E**, the tangent of ϕ may be expressed as;

$$\tan\ \phi=\frac{\mathbf{I}_{\prime\prime}}{\mathbf{I}_{\prime}}=\frac{2\pi f\mathbf{CE}}{\frac{\mathbf{E}}{\mathbf{R}}}=2\pi f\mathbf{CR};$$ from which it is evident,

since **R** is very large numerically, that the numerical value of the angle ϕ is very nearly 90°, and for all practical purposes is considered to be 90°.

CURRENT IN CONDENSERS.

EXAMPLE 42: Find the effective value of the current, in amperes, in a condenser having a capacity of 10 microfarads, (.00001 farad) **if a pressure** (effective) **of** 110 **volts at** 60 **cycles is applied to its terminals.**

The current in this case may be expressed by;

$$I = \frac{E}{\sqrt{-\left(\frac{1}{2\pi fC}\right)^2}} = 2\pi fCE$$

$$= 377 \times 110 \times \frac{10}{1000000} = \mathbf{0.414}$$ **ampere. Answer.**

DATA.

$E = 110$ volts.

$f = 60$ cycles.

$C = \frac{10}{1000000}$

$= .00001$ farad

———O———

PROBLEM 42: Find the effective value of the current, in amperes, in a condenser having a capacity of 5 microfarads (.000005 farad) *if a pressure of 110 volts, effective value, is applied to the terminals of the condenser. The frequency of the pressure being 60 cycles.* **I = 0.207 amperes. Answer.**

Problem 42a: Find the effective value of the current, in amperes, in a condenser having a capacity of 100 microfarads, if an A. C. pressure of 110 volts effective value at 60 cycles, is applied to the terminals of the condenser.

PROBLEM 42b: Find the current in the condenser in **EXAMPLE 42** *using a frequency of 25 cycles in place of 60 cycles. Other values as stated.*
I = 0.086 ampere. (effective). **Answer.**

Problem 42c: Find the *maximum* value of the current under the conditions mentioned in **EXAMPLE 42**, assuming the applied pressure is a sine-curve pressure.

EXAMPLE 43: Compute the energy in the charge of a condenser whose capacity is 100 microfarads, when the pressure between its terminals is 110 volts.

GIVEN DATA.

$C = 100$ mfs $= .0001$ farad.
$E = 110$ volts.

$$J = \tfrac{1}{2}\,CE^2 = \tfrac{1}{2}\ .0001 \times 12100$$
$$= 0.605 \text{ joule; or watt-second.}$$

Compute the number of coulombs in the condenser mentioned in example 43, when charged as stated.

GIVEN DATA.

$J = 0.605$ joule.
$C = .0001$ farad.

$$J = \tfrac{1}{2}\frac{Q^2}{C}\ ;\ \text{from which}$$
$$Q = \sqrt{2JC} = \sqrt{2 \times .605 \times .0001}$$
$$= \mathbf{.011\ coulomb.}$$

PROBLEM 43: Compute the energy of the charge in a condenser whose capacity is 2 microfarads, when the pressure between its terminals is 220 volts.

$J = .0484$ joule or watt-second. Answer.

Problem 43a: Find the value of the capacity of a condenser (in farads or in microfarads) in order that there is one coulomb of electricity in it when the pressure between its terminals is 110 volts.

PROBLEM 43b: Compute the number of coulombs in the condenser mentioned in problem 43 when charged as stated.

$$Q = \sqrt{2JC} = \sqrt{2 \times .0484 \times .000002} = \sqrt{.0000001936} = \mathbf{.00044\ coulomb.}$$

Attention may be directed to the fact that by increasing the pressure applied to a condenser, the necessary capacity of the condenser may be decreased for any desired storage of energy; or of current in the connecting wires leading to the condenser. Advantage is often taken of this fact in practice, to reduce the cost of condensers used in phase-splitters, by using so-called autotransformers that increase the line pressure a considerable amount. In such cases the dielectric used in the condensers must be of such quality as to withstand increased liability of being punctured due to the increased applied pressure from the autotransformers.

EXAMPLE 44: If the sine-curve E. M. F. applied to the terminals of a condenser be changed from + 104 to − 104 volts, in $\frac{1}{250}$ of a second, and the capacity of the condenser is 6.2 microfarads, what is the average rate of current flow during the time? How much energy, expressed in joules, is stored in the condenser at the end of $\frac{1}{100}$ second?

DATA.

$E_{max} = 104$ volts.
$f = 125$ cycles.
$C = 6.2$ mfs.
$= .0000062$ farad.
$2\pi f = 785.4$ (for $f = 125$.)

Since the alternating-pressure changes from one maximum (positive) to another maximum (next negative) in $\frac{1}{250}$ of one second the periodic time **T** of one complete cycle is $\frac{2}{250} = .008 = \frac{1}{125}$ of one second.

The frequency is $\frac{1}{T} = f = 125$ cycles per second.

The maximum value of the current is expressed by:

$$I_{max} = \frac{E_{max}}{\sqrt{\left(-\frac{1}{2\pi f C}\right)^2}} = E_{max} \times 2\pi f C.$$

The value of $2\pi f$ for $f = 125$ cycles may be found on page 210.

The maximum value of the current is therefore:

$$I_{max} = 104 \times 785.4 \times .0000062 = \mathbf{0.5063\ ampere.}$$

If the current is a sine-current, and its maximum value is 0.5063 ampere, its average value is $0.6366 \times 0.5063 = 0.322$ ampere.

Consult page 61 for average value of a sine-curve.

The numerical value of the pressure at the end of $\frac{1}{100}$ second after the beginning of a positive lobe is 104 volts.

The energy, in joules, is expressed by:

$$\begin{aligned} J &= \tfrac{1}{2} E^2 C \\ &= \tfrac{1}{2} \times 104^2 \times .0000062 = \mathbf{.03353\ joules.}\ \text{(nearly)} \end{aligned}$$

This energy is stored during .002 second. If it is assumed that the condenser is charged twice during each complete cycle, the rate of storage will be .06706 joules per second; a small quantity when considered in terms of foot-pounds.*

The current in the wires leading to a condenser, when the condenser is being charged (and discharged) from alternating-current mains, depends upon the *frequency* of the applied pressure. The greater the frequency the larger the current in any given condenser; assuming the applied pressure is constant for all frequencies.

---

*Consult page 6 for equivalents.

PROBLEM 44: If the sine-curve pressure applied to the terminals of a condenser be changed from +104 to −104 volts in $\frac{1}{120}$ of a second, and the capacity of the condenser is 6.2 mfs., what is the average current during this inteval? **0.154 ampere. Answer.**

Problem 44a: Plot a curve, to scale, showing the variation in current in a condenser having a capacity of 10 microfarads, if a constant A. C. pressure of 100 volts (effective value) is applied to the terminals of the condenser; the frequency of the pressure varying from 15 to 125 cycles. Plot points for each increment of 10 cycles. Let 1 inch denote $\frac{1}{10}$ ampere, and 1 inch denote 10 cycles.

EFFECT OF CAPACITY ON INSTANTANEOUS VALUES.

EXAMPLE 45: Find the instantaneous value of the current, in amperes, 7.71 seconds after applying a sine-wave pressure of 110 volts effective, at 60 cycles per second, to the terminals of an arrangement consisting of a non-inductive resistance of 100 ohms connected in series with a condenser whose capacity is 25 microfarads.

The instantaneous value is expressed by:—

$$i = \frac{E_{max}}{\sqrt{R^2 + \left(-\frac{1}{2\pi f C}\right)^2}} \sin\,(2\pi f t + \phi)^*$$

DATA.

$E = 110$ volts (effective).
$E_{max} = 155.5$ volts.
$f = 60\sim$.
$C = 25$ mfs. .000025 farads.
$R = 100$ ohms.
$t = 7.71$ seconds.
$2\pi f = 376.99$ or 377 very nearly.
$2\pi f t = 925.2\,\pi$ or $1.2\,\pi = 216°$.
(See page 59.)

To find the angle of lead ϕ:

$$\tan\phi = -\frac{1}{2\pi f C R} = , -\frac{1}{0.942475}$$

$$= -1.06103.$$

$\phi = 46°42'$ (approximately). The current leads the pressure by 46° 42′.

Substituting the proper values in the general equation gives:—

$$i = \frac{155.5}{\sqrt{21257.2}} \times \sin\,(216° + 46°\,42')$$
$$= \frac{155.5}{146} \times \sin\,(262°\,42')$$
$$= 1.065 \times -\sin\,(262°\,42' - 180°)$$
$$= -1.065 \sin 82°\,42' = -1.065 \times 0.99189$$

= **−1.056 amperes. Answer.**

PROBLEM 45: What will be the instantaneous value of the current in the circuit mentioned in example 45 if the effective value of the pressure is doubled; all other values remaining the same? **−2.112 amperes. Answer.**

---

*The numerical value of the angle ϕ is to be inserted regardless of its sign. The numerical value of the tangent of ϕ being negative merely indicates that ϕ is an angle of lead. The particular instantaneous value designated in this example is very nearly equal to the negative maximum value.

Problem 45a: Find instantaneous value of current in example 45 if the frequency is changed to 25~; all other values remaining unchanged.

In the case cited in example 45, the angle between the pressure $\mathbf{E}_{\mathbf{max}}$ and the horizontal datum is $2\pi ft = 216°$ as shown in figure 51. Here the vector **OA** denotes $\mathbf{E}_{\mathbf{max}} = 155.5$ volts, and the vector **OB** drawn of such length as to denote the maximum value of the current, is located 46° 42′ in *advance* of **OA** ($\mathbf{E}_{\mathbf{max}}$). The equation really expresses the fact that the pressure is to be referred to the horizontal datum, and the current referred to the pressure.

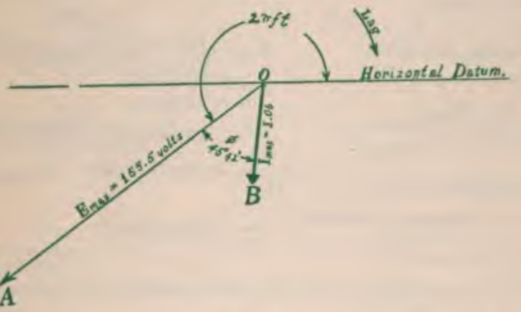

FIG. 51.

It is interesting to note what condition will make i a maximum. The only variable in the second member is an angle; or the sine of an angle. The greatest value a sine may have is 1. Therefore when the

$$\sin(2\pi ft + \phi) = 1, \; i \text{ is a maximum; this occurs when}$$

$$(2\pi ft + \phi) = 90°, 270°, \text{ or any congruent* values.}$$

For $(2\pi ft + \phi) = 270°$, a negative maximum results, and likewise for other *corresponding* congruent values.

---

*See page 59.

MAXIMUM CURRENT VALUES.

EXAMPLE 46: Find the maximum value of the current, expressed in amperes, assuming it to be a sine-wave in form, some time after the application of the pressure, in a circuit whose capacity is $\frac{2}{100000}$ farad, and whose resistance is 10 ohms, when an alternating E. M. F., whose effective value is 110 volts, at 60 cycles per second, is applied to the terminals of the circuit. Find the phase angle of the current, referred to the applied pressure. State whether the angle is one of "lag" or one of "lead".

The maximum current is expressed by:

$$I_{max} = \frac{E_{max}}{\sqrt{R^2 + (-\frac{1}{2\pi f C})^2}}$$

$$= \frac{155.54}{132.6}$$

$= 1.17$ **ampere. Answer.**

DATA.

$E_{eff} = 110$ volts. $E_{max} = 155.54$ volts.

$f = 60$ cycles.

$R = 10$ ohms.

$C = \frac{2}{100000} = .00002$ farad.

$\frac{1}{4\pi^2 f^2} = .0000070362$ (See table, page 209.)

$\frac{1}{C^2} = 2500000000$

If ϕ denotes the angle between the current and the applied pressure then:

$$\tan\phi = -\frac{1}{2\pi f CR} = -.002652 \times 50000 \times \frac{1}{10}$$

$$= -13.26000 \text{ from which is obtained:}$$

$$\phi = -85^\circ\ 41', \text{ and is an angle of } \textit{lead}.$$

The ohmic-component of the current is 85° 41′ in advance of the resultant applied pressure. In reality one component of the applied pressure must be in phase with the ohmic-component of current.

PROBLEM 46: Find the maximum value of the current, expressed in amperes, assuming all conditions as in example 45 except the frequency; which assume to be 25 cycles per second. What is the value of the impedance in example 46? **0.488 amperes. Answer.**

Problem 46a: Find the numerical values of the impedance, reactance, admittance, susceptance and conductance, in **EXAMPLE 46** and in **Problem 46.**

Problem 46b: Plot the results given in **EXAMPLE 46;** adopting a scale of 1 inch = 10 volts, and 1 inch = ½ ampere.

EXAMPLE 47: An alternating sine-wave pressure having a maximum value of 100 volts at 60 cycles is applied to a series circuit. having a resistance of 120 ohms, an inductance of 1 henry and a capacity of 25 microfarads. Find the (effective) **value of the current, in amperes, in the circuit, and its phase angle with reference to the applied pressure.**

The equation applying is:—

$$I_{max} = \frac{E_{max}}{\sqrt{R^2+(2\pi fL-\frac{1}{2\pi fC})^2}}$$

$$= \frac{100}{\sqrt{14400+73386.8}}$$

$$= \frac{100}{\sqrt{87786.8}}$$

$$= \tfrac{100}{296.2}$$

$$= 0.337 \text{ ampere.}$$

$$I_{eff} = .0337 \times 0.70711$$

$$= \mathbf{0.238 \text{ ampere.}} \text{ . Answer.}$$

GIVEN DATA.

$E_{max} = 100$ volts.
$f = 60$ cycles.
$R = 120$ ohms.
$R^2 = 14400$.
$L = 1$ henry.
$C = 25$ mfs.
$= .000025$ farad.

COMPUTED DATA.

$2\pi fL = 377.$
$2\pi fC = .009425.$
$\frac{1}{2\pi fC} = 106.1.$
$\left(2\pi fL - \frac{1}{2\pi fC}\right) = 270.9.$
$\left(2\pi fL - \frac{1}{2\pi fC}\right)^2 = 73436.$

To find the angle between the current and the applied pressure:—

$$\tan\phi = \frac{2\pi fL - \frac{1}{2\pi fC}}{R} = \tfrac{270.9}{120} = 2.2550.$$

$$\phi = \mathbf{66° \ 7'} \text{ and is an angle of } \textit{lag}.$$

In this case $2\pi fL$ is numerically greater than $\frac{1}{2\pi fC}$, so that in subtracting, their difference is positive. If $\frac{1}{2\pi fC}$ in any case is *greater* than $2\pi fL$, then the tangent will be *negative* and the angle will be an angle of

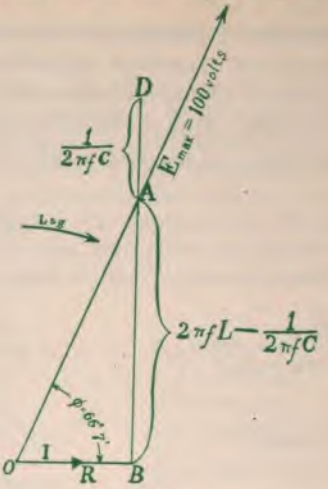

FIG. 52.

lead. This is further illustrated in figure 52, in which the value $\frac{1}{2\pi f C}$ and $2\pi f L$ are shown plotted to scale. In plotting the results of this example the resistance $R = 120$ ohms is taken as the horizontal datum as is customary, and the line **OB** is drawn of proper length to denote 120; from the point **B** the line **BD** is drawn perpendicular to **OB** and of proper length to denote $2\pi f L = 377$; from the point **D** a distance is measured *downward* along **DB** to denote $\frac{1}{2\pi f C}$; then **AB** will denote $2\pi f L - \frac{1}{2\pi f C}$ and if **O** and **A** be connected by a line, then **OA** will represent $\sqrt{R^2 + (2\pi f L - \frac{1}{2\pi f C})^2}$; according to the principle of the "Pythagorean Theorem".

In the triangle **OBA**, the $\tan \phi = \frac{\text{side opp.}}{\text{side adj.}} = \frac{2\pi f L - \frac{1}{2\pi f C}}{R}$. From the figure it may be seen if **DA** should be longer than **DB** then **OA** will fall *below* the horizontal datum, the current vector **I** would *"lead"* the pressure vector (along **OA**), and the angle ϕ would be called an *angle of lead.*

PROBLEM 47: An alternating sine-wave pressure, having an effective value of 70.7 volts at *25 ∼ is applied to a circuit, having a resistance of 120 ohms, an inductance of 1 henry and a capacity of .000025 farad: find the effective value of the current, in amperes, and its phase angle referred to the applied pressure as datum.* **0.456 amperes. Answer.**

Problem 47a: Find the numerical value of **X**, **Y**, **Z**, b, and g; that is, respectively:—reactance, admittance, impedance, susceptance, and conductance in *example* 47 and *problem* 47.

$$\text{Reactance, } \mathbf{X} = \left(2\pi f\mathbf{L} - \frac{1}{2\pi f\mathbf{C}}\right) = \mathbf{Z}\sin\phi =$$

$$\text{Impedance, } \mathbf{Z} = \sqrt{\mathbf{R}^2+\mathbf{X}^2} =$$

$$\text{Admittance, } \mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{1}{\sqrt{\mathbf{R}^2+\mathbf{X}^2}} =$$

$$\text{Susceptance, } b = \frac{\mathbf{X}}{\mathbf{Z}^2} = \mathbf{Y}\sin\phi =$$

$$\text{Conductance, } g = \frac{\mathbf{R}}{\mathbf{Z}^2} = \mathbf{V}\cos\phi =$$

CAPACITIES OF CONDENSERS DEPENDING ON THEIR ARRANGEMENT.

EXAMPLE 48: Given two condensers having respective capacities of 10 and 20 microfarads; find their combined capacity when connected together in series, and when connected together in parallel.

The combined capacity of two or more condensers may easily be computed by remembering it is to be found in a manner exactly *opposite* to the computing of the joint or equivalent resistance of two or more resistances in "direct-currents".

DATA.

$C_{,} = 10$ microfarads.

$C_{,,} = 20$ microfarads.

When two condensers are connected together in parallel they really constitute *one condenser*, having a number of plates equal to the sum of the number of plates in the separate condensers.

When condensers are connected together in *parallel*, their combined capacity is equal to the sum of the capacities of the individual condensers; corresponding to resistances in *series*. The combined capacity of the two given condensers is $10+20=$ **30 mfs. Answer.**

When two or more condensers are connected together in series, one having a capacity denoted by $C_{,}$ and another having a capacity denoted by $C_{,,}$, their combined capacity will be denoted by;

$$C = \frac{C_{,}\,C_{,,}}{C_{,,}+C_{,}}\,. \quad \text{In the given case } C = \frac{10\times20}{20+10} = 6\tfrac{2}{3} \text{ mfs. Answer.}$$

PROBLEM 48: Find the combined or equivalent capacity of three condensers, having respectively 1, 2 and 3 microfarads capacity when all are connected together in series and when all are connected together in parallel.

$$\text{In series } C = \frac{C_{,}C_{,,}C_{,,,}}{C_{,,}C_{,,,}+C_{,}C_{,,,}+C_{,}C_{,,}} = \frac{6}{11} = 0.5454 \text{ etc. microfarads.}$$

In parallel $C =$ **6 microfarads. Answer.**

Problem 48a: Find the combined capacity of four condensers having respectively, 1, 2, 3, and 4 microfarads when connected together in series, and also when connected together in parallel.

EXAMPLE 49: Find the combined equivalent capacity of the arrangement shown in figure 53 with the numerical values there given.

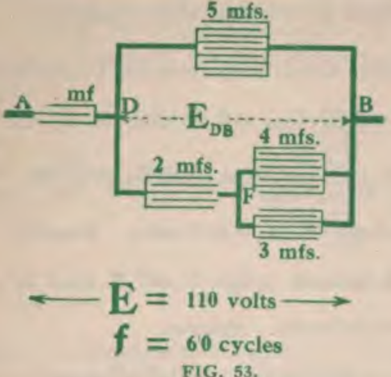

FIG. 53.

First combine 3 and 4 (in parallel) giving; $3+4=7$ microfarads.
Then combine 7 and 2 (in series) giving; $\frac{14}{9}=1\frac{5}{9}=1.555$ mfs.
Next combine $\frac{14}{9}$ and 5 (in parallel) giving; $\frac{14}{9}+\frac{45}{9}=\frac{59}{9}=6.55$ mfs.
Last combine $\frac{59}{9}$ and 1 (in series) giving; $\frac{59}{68}=0.867$ mfs.

PROBLEM 49: If in figure 53 the capacity between **A** *and* **D** *had been* $\frac{1}{3}$ *mf.; between* **D** *and* **F** 2 *mfs.; between* **F** *and* **B**, 4 *and* 1 *mfs. respectively, and between* **D** *and* **B** 3 *mjs., the combined capacity would have been*

0.31 mfs. Answer.

Problem 49a: Given three condensers having respective capacities of 1, 2 and 3 microfarads, find all possible values of capacity, by employing the given condensers.

EXAMPLE 50: If a constant alternating-pressure of 100 volts, at 60 cycles, should be applied to the terminals of the arrangement shown in figure 53 find the numerical value, in volts, of the pressure between the terminals of each individual capacity.

The total or resultant current in this case will be expressed by:

$\mathbf{I} = 2\pi f\mathbf{CE}$; also $\mathbf{I} = 2\pi f\mathbf{C}_1\mathbf{E}_{\mathbf{AD}}$ and dividing one by the other gives;

$$1 = \frac{2\pi f\mathrm{CE}}{2\pi f\mathrm{C}_1\mathrm{E}_{\mathrm{AD}}} \text{ from which } \mathbf{E}_{\mathbf{AD}} = \frac{\mathbf{C}}{\mathbf{C}_1}\mathbf{E}$$

From which $\mathbf{E}_{\mathbf{AD}} = 0.867 \times 100 =$ **86.7 volts. Answer.**

The pressure $\mathbf{E}_{\mathbf{DB}}$ between points **D** and **B** must be

$100 - 86.7 =$ **13.3 volts. Answer.**

Current $\mathrm{I}_5 = 2\pi f\mathrm{C}_5 13.3$ and $\mathrm{I}_2 = 2\pi f\mathrm{C}_2\mathrm{E}_{\mathrm{DF}}$, and

$$\frac{\mathrm{I}_5}{\mathrm{I}_2} = \frac{2\pi f\mathrm{C}_5 \times 13.3}{2\pi f\mathrm{C}_2 \times \mathrm{E}_{\mathrm{DF}}}.$$

But $\mathrm{I}_2 = 2\pi f\mathrm{C}_{\mathrm{X}}\mathrm{E}_{\mathrm{DB}}$ and $\mathrm{I}_5 = 2\pi f\mathrm{C}_5\mathrm{E}_{\mathrm{DB}}$.

Then $\dfrac{2\pi f\mathrm{C}_5\mathrm{E}_{\mathrm{DB}}}{2\pi f\mathrm{C}_{\mathrm{X}}\mathrm{E}_{\mathrm{DB}}} = \dfrac{\mathrm{C}_5 \times 13.3}{\mathrm{C}_2 \times \mathrm{E}_{\mathrm{DF}}}$; from which

$$\mathrm{E}_{\mathrm{DF}} = \frac{\mathrm{C}_{\mathrm{X}}}{\mathrm{C}_2} \times 13.3 = \frac{1\frac{5}{9}}{2} \times 13.3 = \mathbf{10.34\ volts.}$$

The pressure $\mathrm{E}_{\mathrm{FB}} = 13.3 - 10.34 =$ **2.96 volts. Answer.**

PROBLEM 50: If a constant pressure of 300 volts is applied to the terminals of the arrangement mentioned in problem 48, find the numerical value of the pressure across each individual portion of the arrangement.

$\mathbf{E}_{\mathbf{AD}} =$ **278 volts;** $\mathbf{E}_{\mathbf{DB}} =$ **21 volts.**
$\mathbf{E}_{\mathbf{DF}} =$ **15 volts;** $\mathbf{E}_{\mathbf{FB}} =$ **6 volts. Answer.**

Problem 50a: If a constant pressure of 300 volts at 25 cycles is applied to the terminals of the arrangement shown in figure 53 page 171, find the numerical values of the pressures across each individual portion.

EXAMPLE 51: Given a coil having a resistance R of 10 ohms, and a coefficient of inductance L = 0.18 henry, connected in *parallel* with a condenser having a capacity of 10 microfarads. If a pressure of 110 volts at 60 cycles is applied to the terminals of the arrangement find the value of the main line current, in amperes, and its phase angle with reference to the applied pressure.

DATA.

$E = 110$ volts.
$f = 60 \sim$
$C = 10$ mfs. $= \frac{1}{100000}$ farad.
$R_{,,} = 10$ ohms.
$L_{,,} = 0.18$ henry.
$2\pi f = 377.$
$2\pi f L_{,,} = 67.86$

———O———

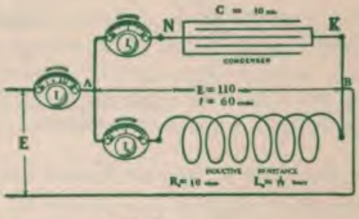

FIG. 54.

Figure 54 shows the arrangement of apparatus, and figure 55 is a vector diagram showing the relations of the applied pressure, and the currents. The ammeter denoted by **I** indicates the effective value, in amperes, of the main line current; while the ammeter denoted by $\mathbf{I}_{,}$ indicates the effective value, in amperes, of the current in the *condenser* branch, and ammeter $\mathbf{I}_{,,}$ indicates the effective values in amperes of the current in the coil; or in the *inductive* branch.

The current in each branch circuit will be found first; with its phase relation with the applied pressure, and the value of the *resultant* of the two branch currents will then be computed and its phase relation with the applied pressure ascertained.

The applied pressure **E** is common to both branches.

The current in the coil is expressed by:

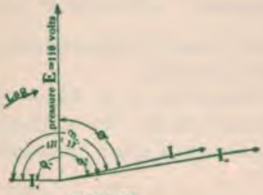

FIG. 55.

$\mathbf{I}_{,} = \frac{E}{\sqrt{R^2 + (2\pi f L)^2}}$. Substituting in this equation, the proper numerical values the following is obtained:

$$\mathbf{I}_{,} = \frac{110}{\sqrt{100 + (377 \times 0.18)^2}} = \frac{110}{\sqrt{100 + (67.86)^2}} = \frac{110}{\sqrt{4704.9796}} = \frac{110}{68.59}$$

$= 1.6$ ampere.

The angle $\phi_{,}$ between $\mathbf{I}_{,}$ and the applied pressure **E** is obtained from:

$$\tan\phi_{,} = \frac{2\pi f\mathrm{L}}{\mathrm{R}} = \frac{67.86}{10} = 6.786$$

Therefore $\phi_{,} = 81° \, 37'$.

The current in the condenser is expressed by.:

$$\mathbf{I}_{,,} = \frac{\mathrm{E}}{(-\frac{1}{2\pi f\mathrm{C}})^2} = 2\pi f\mathrm{CE}.$$

Substituting the proper numerical values in the equation will give:

$$\mathbf{I}_{,,} = \frac{377 \times 1 \times 110}{100000} = 0.414 \text{ ampere.}$$

The angle $\phi_{,,}$ between $\mathbf{I}_{,,}$ and the applied pressure **E** is obtained from:

$$\tan\phi_{,,} = -\frac{1}{2\pi f\mathrm{CR}} = -\infty \qquad (\mathrm{R}_{,} = \mathrm{O})$$

Therefore $\phi_{,,} = -90°$ being an angle of *lead*, the current $\mathbf{I}_{,,}$ is 90° in *advance* of **E**. See figure 54.

The angle between $\mathbf{I}_{,}$ and $\mathbf{I}_{,,}$ must therefore be $90° + 81° \, 37' = 171° \, 37'$. See figure 55.

The resultant current, or the current in the main line, is expressed by:

$$\mathbf{I} = \sqrt{\mathbf{I}_{,}^2 + \mathbf{I}_{,,}^2 + 2\mathbf{I}_{,}\mathbf{I}_{,,} \cos 171° \, 37'}.$$

DATA.

$\mathbf{I}_{,}^2 = .1713$

$\mathbf{I}_{,,}^2 = 2.56$

$\cos 171° \, 37' = -0.98931$

Substituting the proper numerical values in the foregoing equation gives:

$$\mathbf{I} = \sqrt{.1713 + 2.56 - 1.3106}$$

$= \sqrt{1.4207} =$ **1.19 ampere.** This value is *less* than the value of $\mathbf{I}_{,,}$ as may be seen in figure 55, page 173, illustrating a case in which the line current is less than a branch current.

The angle of phase denoted by ϕ in figure 55 between the assumed datum of the applied pressure **E** and the *main-line current* **I** is found as follows: The angle β between **I** and $\mathbf{I}_{,,}$ is obtained from:

$$\cos\beta = -\frac{\mathbf{I}_{,,}^2 - \mathbf{I}_{,}^2 - \mathbf{I}^2}{2\mathbf{I}_{,}\mathbf{I}} = -\frac{2.56 - 0.1713 - 1.4207}{2 \times 0.414 \times 1.19} = -\frac{0.968}{0.98532} = -0.98265$$

From which $\beta = 169° \, 19'$.

The angle ϕ, the angle between the main line current **I** and the applied pressure **E** as indicated in figure 55 is $= \beta - \phi_{,} = 169° \, 19' - 90° = 79° \, 19'$.

PROBLEM 51: If a coil having a resistance of 10 *ohms and a coefficient of inductance of* $\frac{2}{10}$ *henry is connected in parallel with a condenser having a capacity of* 2.6 *microfarads, and a pressure of* 104 *volts at* 60 *cycles is applied to the terminals of the arrangement, like figure* 54, *page* 173, *compute the value of the main line current, in amperes, and ascertain its phase angle relative to the applied pressure.* **0.983 ampere. Answer.**

Problem 51a: Given a coil having a resistance of 100 ohms and a coefficient of inductance of 10 henrys, connected in parallel with a condenser having a capacity of $\frac{7}{10}$ microfarad. Compute the value of the main line current, in amperes, and find its phase angle with reference to the applied pressure, if a pressure of 110 volts at 60 cycles is applied to the arrangement.

Problem 51b: If the frequency of the applied pressure in example 50 page 172 is changed to 25 cycles, find the numerical value of the main line current, in amperes, and its phase with reference to the applied pressure. Other data as given.

Problem 51c: Solve problem 50 page 172, changing the frequency to 15 cycles. All other data as given.

Problem 51d: Solve problem 50a, page 172, changing the frequency to 125 cycles. Other data as given.

Problem 51e: Draw a vector diagram to scale of 1 inch = 10 volts and 1 inch = $\frac{1}{5}$ ampere, showing the relation of the currents and the various pressure components in problem 50a, page 172.

TO COMPUTE CURRENT IN ANY ARRANGEMENT.

Under the ordinary conditions of constant pressure or constant potential supply, the common and practical problems are to find the current or currents in given single inductive circuits or in any arrangement of inductive, non-inductive, and condensive circuits.

By "*any arrangement*" is meant that the inductances, resistances and capacities may be connected with one another either in simple series connections, in multiple, or that two or more series arrangements may be connected together in parallel, and in addition may be connected in series with other resistances, inductances and capacities.

Figure 56, page 177, is a diagramatic representation of a simple series arrangement; figure 58, page 179 denotes a parallel arrangement of the same three portions, while figure 69, page 195, is a representation of a more complex arrangement of several resistances, inductances and capacities.

While the examples and the problems given here under the different arrangements may refer to simple resistances, inductances and capacities as such, one should not lose sight of the fact they really represent practical conditions, as useful operative devices or apparatus, such as motors, transformers, wireless telegraph devices, induction coils, transmission lines and generators.

To make the discussion and explanation of the various arrangements easily comprehended, the arrangements will be discussed in the order of complexity; using the same single portions in as many arrangements as possible, beginning with the series arrangement.

EXAMPLE 52: Suppose a coil having a resistance of 10 ohms and a coefficient of inductance $L=\frac{1}{10}$ henry, a condenser having a capacity of 100 microfarads (= .000100 farad) and a non-inductive resistance of 10 ohms are all connected together in series, and a sine-pressure of 110 volts at 60 cycles is applied to the terminals of the complete arrangement; find the numerical value of the current in the circuit, and the numerical values and the phase relations of the component pressures existing between the terminals of each separate portion together with their phase angles relative to the applied pressure of 110 volts.

Figure 56 illustrates the arrangement of the several portions, and figure 57 shows the relation of the various vector quantities.

GIVEN DATA.

$E=110$ volts.
$f=60$ cycles.
$R=10$ ohms. $R_L=10$ ohms.
$L=\frac{1}{10}$ henry.
$C=100$ mfs. $=.0001$ farads.

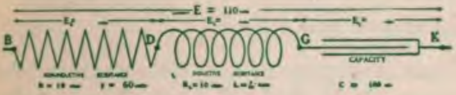

FIG. 56.

The current in the circuit may be expressed by:

$$I=\frac{E}{\sqrt{(R+R_L)^2+(2\pi fL-\frac{1}{2\pi fC})^2}}$$

$$=\frac{110}{\sqrt{(10+10)^2+(37.7-\frac{1}{.0377})^2}}$$

$$=\frac{110}{\sqrt{400+125.44}}=\frac{110}{22.9}$$

$$=\textbf{4.8 amperes.}$$

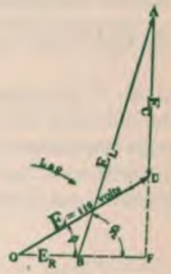

FIG. 57.

The angle ϕ between this current and the applied pressure may be found from:

$$\tan\phi=\frac{2\pi fL-\frac{1}{2\pi fC}}{R+R_L}=\frac{37.7-26.5}{10+10}$$

$$=\frac{11.2}{20}=0.56$$

From which $\phi=29°\ 15'$

The pressure component across the non-inductive section is

$= RI = 10 \times 4.8 = 48$ volts, and is in phase with the current.

The pressure component across the inductive portion is

$I\sqrt{R_L^2+(2\pi fL)^2} = 39.0 \times 4.8 = 187.2$ volts, and the angle ϕ_L between it and the current is found from:

$$\tan\phi_L = \frac{2\pi f L}{R_L} = \frac{37.7}{10} = 3.77$$

From which $\phi_L = 75° 9'$.

The pressure component across the capacity section is =

$\frac{I}{2\pi fC} = \frac{4.8}{.0377} = 4.8 \times 26.5 = 127.3$ volts, and its phase angle ϕ_C is found from:

$$\tan\phi_C = -\frac{1}{2\pi fC \times 0} = -\infty$$

From which $\phi_C = -90°$; an angle of lead.

A *vector* diagram drawn to scale, showing the pressure components and the applied pressure is given in figure 57; in which **OB** denotes $RI = 48$ volts; **BA** denotes the pressure (between the terminals of the inductive portion of the circuit) $I\sqrt{R^2+(2\pi fL)^2}$; **AD** denotes the pressure $\frac{I}{2\pi fC}$, between the terminals of the condenser; **BF** denotes the ohmic pressure-component of the inductive portion expressed by $R_L I$, and **AF** denotes the reactive pressure-component (of the jnductive portion) expressed by $2\pi fLI$.

The arithmetic sum of E_R, E_L and E_C is; $48+187.2+127.3=362.5$; more than three times their vectorial resultant of 110 volts.

EXAMPLE 52a: Suppose a coil having a resistance of 10 ohms and a coefficient of inductance L $= \frac{1}{10}$ henry, a non-inductive resistance of 10 ohms, and a condenser having a capacity of 100 mfs. (= .000100 farads) are all connected together in parallel, and a pressure of 110 volts at 60 cycles is applied to the arrangement; find the numerical value of the current in the main line supplying the arrangement, the numerical values of the various component currents, and their phase relation to the applied pressure.

Figure 58 shows the arrangement.

Consult page 120.

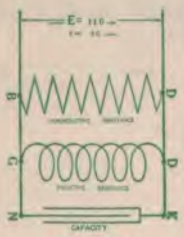

FIG. 58.

GIVEN DATA.

$E = 110$ volts.

$f = 60$ cycles.

$R = 10$ ohms. $R_L = 10$ ohms.

$L = \frac{1}{10}$ henry

$C = 100$ mfs. $=$ .0001 farad.

———O———

The current in the non-inductive portion is $= \frac{110}{10} = 11$ amperes and is in phase with the applied pressure.

The current in the inductive section is $= \frac{110}{39} = 2.8$ amperes.

The angle between this current and the applied pressure is found from

$$\tan \phi_L = \frac{2\pi f L}{R_L} = 3.77, \text{ and } \phi_L = 75° \, 9'.$$

The current in the capacity section is $= 2\pi f CE = \frac{377 \times 110}{10000} = 4.1$ amperes, and leads the applied pressure by 90°; or $\phi_C = -90°$.

The resultant or main line current, **I**, may be found by finding the *vectorial resultant* of the three currents just obtained; or by the method outlined on page 144. Each method serves as a check on the other.

The vector diagram of currents is shown in figure 59. By the method of projections, using the applied pressure line as the **X** axis and the capacity current line as the **Y** axis, the following is true;

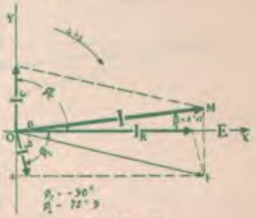

FIG. 59.

X projections.		Y projections.	
$\mathbf{I_R} \cos 0^\circ =$	11.00	$\mathbf{I_R} \sin 0^\circ =$	0.00
$\mathbf{I_L} \cos 75^\circ\ 9' =$	.71	$\mathbf{I_L} \sin 75^\circ\ 9' =$	2.71
$\mathbf{I_C} \cos -90^\circ =$	0.00	$\mathbf{I_C} \sin -90^\circ =$	-4.1
Sum of the X projections =	11.71	Sum of the Y projections =	-1.39

From which; $\mathbf{I} = \sqrt{11.71^2+(-1.39)^2} =$ **11.79 amperes. Answer.**

and $\tan\phi = \frac{-1.39}{11.71} = -0.118$; therefore $\phi = -6^\circ\ 47'$; showing that the resultant current *leads* the applied pressure, at an angle of 6° 47′.

It may be noted that the sum of the three currents is 17.9 while their vectorial resultant is only 11.79.

EXAMPLE 52b: If a coil having a resistance of 10 ohms and a coefficient of inductance $L = \frac{1}{10}$ henry, is connected in parallel with a condenser having a capacity of 100 mfs.; the two connected in series with a non-inductive resistance of 10 ohms, and a pressure of 110 volts at 60 cycles is applied to the terminals of the arrangement, find the numerical value of the current in the main supplying the arrangement and its phase relation with the applied pressure.

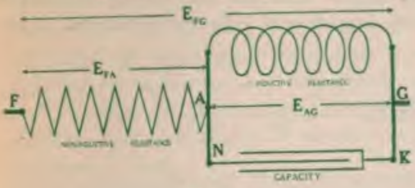

FIG. 60.

GIVEN DATA.

$E_{FG} = 110$ volts.
$f = 60$ cycles.
$R = 10$ ohms; $R_L = 10$ ohms.
$L = \frac{1}{10}$ henry.
$C = 100$ mfs. $= .0001$ farad.

COMPUTED DATA.

$2\pi f = 377$
$2\pi f L = 37.7.$
$\frac{1}{2\pi f} = .002652$ (See page 209.)
$\frac{1}{2\pi f C} = 26.5.$

Figure 60 shows the arrangement of the apparatus and figure 61 shows the vector relation of the various pressures, currents and admittances.

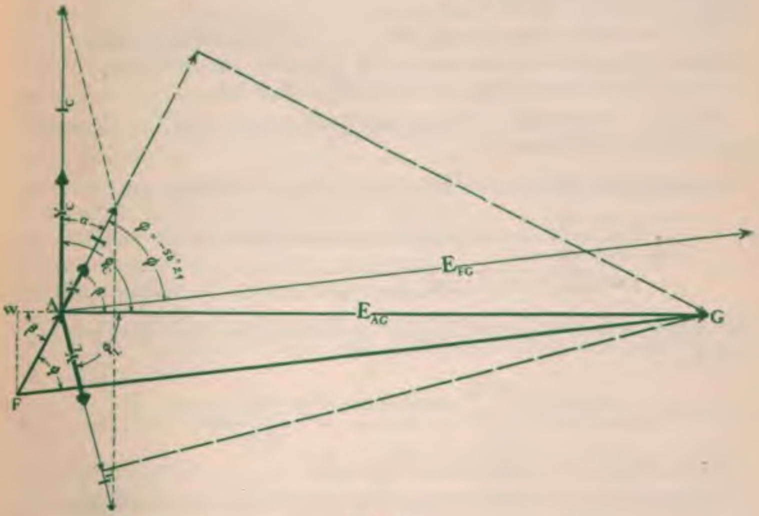

FIG. 61.

The current in the inductive branch will be denoted by $\mathbf{I_L}$; the current in the condenser by $\mathbf{I_C}$, and the resultant current (the current in the non-inductive portion) by **I**.

The pressure component $\mathbf{E_{AG}}$ between **A** and **G** figure 60 is common to both the inductive branch and the branch containing the capacity. The current $\mathbf{I_L}$ multiplied by the impedance of the inductive portion, will give the numerical value of $\mathbf{E_{AG}}$. Also the current $\mathbf{I_C}$ multiplied by the impedance of the capacity branch will give the numerical value of $\mathbf{E_{AG}}$. This means that

$$\mathrm{E_{AG}} = \mathrm{I_L}\sqrt{\mathrm{R_L}^2+(2\pi f\mathrm{L})^2} = \mathrm{I_C}\sqrt{-\left(\frac{1}{2\pi f\mathrm{C}}\right)^2}.$$

The unfortunate fact of the matter is that both the component currents and $\mathbf{E_{AG}}$ are unknown. It is in order then to find the value either of the component currents; of their resultant, or of $\mathbf{E_{AG}}$.

It is to be observed that the applied pressure $\mathbf{E_{FG}}$ has at least two components; $\mathbf{E_{AG}}$ and $\mathbf{E_{FA}}$ ($=\mathrm{RI}$) which must combine *vectorially*.

It would facilitate the mathematical processes to find if possible, expressions for $\mathbf{E_{AG}}$ and $\mathbf{E_{FA}}$ in terms of those quantities whose values are known or easily found.

The joint admittance of the inductive and condenser portions, multiplied by the pressure component $\mathbf{E_{AG}}$ will give the resultant current, **I**, which is in phase with the non-inductive portion **FA**.

The joint admittance **Y** of the inductive portion and the condenser, may be found as follows:

Letting $\mathbf{Y_L}$ denote the *admittance* of the inductive portion, and

$\mathbf{Y_C}$ denote the admittance of the condensive portion, then

$\mathrm{Y}=\sqrt{\mathrm{Y_L}^2+\mathrm{Y_C}^2+2\mathrm{Y_L}\,\mathrm{Y_C}\,\cos(\phi_\mathrm{L}+\phi_\mathrm{C})}$ in which ϕ_L denotes the angle between $\mathbf{E_{AG}}$ and $\mathbf{I_L}$ and ϕ_C denotes the angle between $\mathbf{E_{AG}}$ and $\mathbf{I_C}$.

$$\tan\phi_\mathrm{L}=\frac{2\pi f\mathrm{L}}{\mathrm{R}},\ \text{and}\ \tan\phi_\mathrm{C}=-\frac{1}{2\pi f\mathrm{C}\times 0}=-\infty.$$

From which $\phi_\mathrm{L} = 75^\circ\,9'$, and $\phi_\mathrm{C}=-90^\circ$.

Therefore $\phi_\mathrm{L}+\phi_\mathrm{C}=165^\circ\,9'$, and $\cos 165^\circ\,9' = -0.96660$; being the [illegible]e of 14° 51′ with a negative sign.

The numerical value of **Y** is found by substituting

$$\mathbf{Y_L} = \frac{1}{\mathbf{Z_L}} = \frac{1}{39} = .0256, \text{ and}$$

$$\mathbf{Y_C} = \frac{1}{\mathbf{Z_C}} = \frac{1}{\left(-\frac{1}{2\pi fC}\right)} = -2\pi f C = -.0377; \quad \text{giving:}$$

$$\mathbf{Y} = \sqrt{.0256^2 + .0377^2 - 2 \times .0256 \times .0377 \times (-0.9666)}$$
$$= 0.0145217.$$

In order to make more intelligible the fundamental principles governing problems of this nature, the following method of procedure is outlined. A short study of this particular example, will show how to apply the different equations given, in solving problems of a similar nature, in a minimum of time.

It is not essential nor necessary that the various vectors or lines in the diagram shown in figure 61 have *numerical* values assigned to them in discussing the various trigonometrical relations of the various portions of the diagram. While the diagram in figure 61 is drawn to scale, no regard will be paid to this at present.

Having found the numerical values of the admittances $\mathbf{Y_L}$, $\mathbf{Y_C}$ and **Y** and of the angles ϕ_Z and ϕ_C with the line **AG** chosen as horizontal datum, the *magnitudes and directions* of the *admittances* may be properly represented as shown. It is seen that the *angular relation* of the admittances is the same as the angular relation of the corresponding currents.

The symbols shown in figure 62 will be adopted in connection with the vector diagrams, to render interpretation easier.

The angle between the component pressure $\mathbf{E_{AG}}$, denoted by the horizontal line **AG**, and the condenser admittance is 90°. $\mathbf{Y_C} = -.0377$ is therefore drawn perpendicular to **AG** and *leading* it by 90°. The angle between the component pressure $\mathbf{E_{AG}}$ and the admittance of the inductive branch is 75° 9′, the vector denoting the inductive admittance is drawn as $\mathbf{Y_L} = .0256$ and lagging **AG** by 75° 9′. It is evident that angle α, the angle between $\mathbf{Y_C}$ and **Y** may be obtained from;

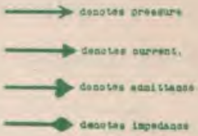

FIG. 62.

$$\cos\alpha = \frac{\mathbf{Y_C}^2 + \mathbf{Y}^2 - \mathbf{Y_L}^2}{2\mathbf{Y_C}\mathbf{Y}},$$ and that the value of angle β may be found from $90° - \alpha$. It is important to know the value of β because of its trigonometrical relations with other quantities to be considered. In the given example the numerical value of β is found to be 63° 19′.

It is important to note that the pressure between the terminals of the non-inductive portion of the circuit is in phase with the resultant current, and should therefore be shown in line with the resultant current vector in the diagram. The resultant current vector must also be in phase with the joint admittance vector as indicated.

Suppose the line **AF** denotes the pressure **RI** between the terminals of the non-inductive portion, assuming it to be of any arbitrary length whatever, if the line **AG** is extended toward the left, and a perpendicular, **WF** drawn to it from the point F, the following relations are true:

$$\mathbf{WA} = \mathbf{RI} \cos \beta, \text{ and } \mathbf{WF} = \mathbf{RI} \sin \beta.$$

While in the case under consideration the numerical values of the pressures $\mathbf{E_{FA}}$ and $\mathbf{E_{AG}}$ are unknown, the relations just mentioned hold true, and from these relations $\mathbf{E_{FA}}$ and $\mathbf{E_{AG}}$ may be calculated.

From the vector diagram it may be seen that

$$\overline{\mathbf{FG}}^2 = \overline{\mathbf{WG}}^2 + \overline{\mathbf{WF}}^2; \quad \mathbf{WA} = \mathbf{RI} \cos \beta, \text{ and } \mathbf{WF} = \mathbf{RI} \sin \beta.$$

But **FG** denotes the applied pressure $= \mathbf{E_{FG}} = 110$ volts.

Then $110 = \sqrt{\overline{\mathbf{WG}}^2 + \overline{\mathbf{WF}}^2} = \sqrt{(\mathbf{WA}+\mathbf{AG})^2 + \overline{\mathbf{WF}}^2}$

But **AG** denotes $\mathbf{E_{AG}}$; the unknown pressure component, and after making the proper substitutions the following results:

$$110 = \sqrt{\mathbf{R^2I^2} \cos^2 \beta + 2\mathbf{E_{AG}}\,\mathbf{RI} \cos \beta + \mathbf{E_{AG}}^2 + \mathbf{R^2I^2} \sin^2 \beta}$$

This may be simplified by remembering that $\sin^2 \beta + \cos^2 \beta = 1$,

to; $$110 = \sqrt{\mathbf{R^2I^2} + 2\mathbf{E_{AG}}^2 \times \mathbf{RI} \cos \beta + \mathbf{E_{AG}}^2}$$

There are two unknowns in this equation, **I** and $\mathbf{E_{AG}}$.

It is known however that $\mathbf{I} = \mathbf{E_{AG}Y}$, and substituting this value in the last equation gives;

$110 = \sqrt{\mathbf{E_{AG}}^2\mathbf{Y^2R^2} + 2\mathbf{E_{AG}}^2\mathbf{YR} \cos \beta + \mathbf{E_{AG}}^2}$; which may be expressed as;

$110 = \mathbf{E_{AG}}\sqrt{\mathbf{Y^2R^2} + 2\mathbf{YR} \cos \beta + 1}$; in which the numerical value of all expressed quantities are known except $\mathbf{E_{AG}}$.

Solving for $\mathbf{E_{AG}}$ gives;

$$\mathbf{E_{AG}} = \frac{110}{\sqrt{\mathbf{Y^2R^2} + 2\ \mathbf{YR} \cos\beta + 1}}$$, which applies to any similar case where the values of the applied pressure, joint admittance, non-inductive resistance and cos β are known.

Substituting the proper numerical values of the various quantities given;

$\mathbf{E_{AG}} = 102.61$ volts.

The numerical value of the resultant current is found from

$\mathbf{I} = \mathbf{E_{AG}Y} = 102.61 \times .01452 = 1.48$ amperes, or nearly 1.49 amperes.

Another mothod of obtaining the numerical value of the resultant current is as follows:

Since the resultant current $\mathbf{I} = \mathbf{E_{AG}Y}$, is in phase with the pressure $\mathbf{E_{FA}}$ between the terminals of the non-inductive portion of the arrangement, $\mathbf{E_{FA}} = \mathbf{RI} = \mathbf{RE_{AG}Y}$. Substituting the proper numerical values gives;

$$\mathbf{E_{FA}} = 10 \times 102.61 \times .01452 = 14.87 \text{ volts.}$$

The resultant current, **I**, is found from;

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{R}} = \frac{14.87}{10} = \mathbf{1.48\ ampere.} \qquad \mathbf{Answer.}$$

The angle ϕ between this current and the applied pressure **E**, may be found from;

$$\cos\phi = -\frac{\mathrm{E_{AG}}^2 - \mathrm{E_{FA}}^2 - \mathrm{E_{FG}}^2}{2\mathrm{E_{FA}}\ \mathrm{E_{FG}}} = -0.5539;$$ from which $\phi = \mathbf{56^\circ\ 24'}$; an angle of lead. The main line current, leads the pressure applied to the *arrangement* by the angle stated.

The important relations applying to cases similar to the one just explained are given as follows:

Joint admittance $\mathbf{Y} = \sqrt{\mathrm{Y_L}^2 + \mathrm{Y_C}^2 + 2\mathrm{Y_L}\mathrm{Y_C} \cos(\phi_\mathrm{L} + \phi_\mathrm{C})}$

$$\cos\alpha = \frac{\mathrm{Y}^2 + \mathrm{Y}^2 - \mathrm{Y_L}^2}{2\mathrm{Y_C}\mathrm{Y}}$$

$$\cos\beta = \cos(90^\circ - \alpha)$$

Pressure component between the terminals of the parallel portion is;

$$E_{AG} = \frac{\text{Applied pressure in volts}}{\sqrt{Y^2R^2 + 2YR\cos\beta + 1}}$$

Resultant current is equal to;

$$I = E_{AG}Y$$

Pressure component between the terminals of the non-inductive portion is

$$E_{FA} = RI = RE_{AG}Y.$$

Angle of lag of the resultant or main line current **I** behind the applied pressure E_{FG} is found from;

$$\cos\phi = -\frac{E_{AG}^2 - E_{FA}^2 - E_{FG}^2}{2E_{FA}E_{FG}}.$$

In constructing the vector diagram from which figure 61 is a reduced copy, 1 inch denoted 10 volts; impedance same scale; 1-2 inch denoted 1 ampere, and $\frac{1}{10}$ inch denoted an admittance of .01.

The triangle **FAG** denotes a triangle of pressures; the *direction* of the pressure components is to be noted. The vector pressure **FA** points upward; the vector pressure **AG** points to the right; while the resultant of these two vector pressure points *against* them. That is in passing over the vector pressures in a clock-wise direction starting at point **F**, the direction of the two pressure vectors **FA** and **AG** will be with the direction of motion while the direction of the resultant pressure vector **FG** will be opposite to that of the direction of motion around the pressure triangle. The triangle of pressure vectors corresponds with a polygon of forces in Mechanics.

The three pressure vectors constituting the triangle **FAG** might all be drawn from the single point **A** which might make it clearer that the pressure E_{FG} is the resultant of E_{FA} and E_{AG}.

Problem 52b: Construct a vector diagram to scales given in example 52b, showing relations of pressures, currents, and .admittances in an arrangement as given in example 52b; with all values there given except the frequency; change this to 25 cycles.

$E_{FA} = 19.53$ *volts.*

$E_{AG} = 95.77$ *volts.*

$I = 1.95$ *ampere.*

$\phi = 38° \ 30'.$

EXAMPLE 52c: If a coil having a resistance of 10 ohms and a coefficient of inductance $L = \frac{1}{10}$ henry, is connected in parallel with a non-inductive resistance of 10 ohms, the two connected in series with a condenser having a capacity of 100 mfs., and a pressure of 110 volts at 60 cycles applied to the terminals of the arrangement, find the numerical value of the current in the condensed portion, and its phase angle with reference to the applied pressure.

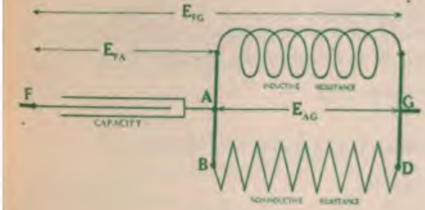

FIG. 63.

GIVEN DATA.

$E = 110$ volts.
$f = 60$ cycles.
$R = 10$ ohms; $R_L = 10$ ohms.
$L = \frac{1}{10}$ henry.
$C = 100$ mfs. $= .0001$ farad.

Figure 63 shows the arrangement, and figure 64 shows the vector relation of the various pressures, currents and admittances. The important data are tabulated and when coordinated with the diagrams, according to the method outlined in example 51b, no lengthy explanation is necessary.

In the vector diagram from which figure 64 is a reduced copy, 1 inch denoted 10 volts; impedances same scale; $\frac{1}{2}$ inch denoted 1 ampere; 10 inches denoted 1 unit of admittance, or an admittance of $.01 = \frac{1}{10}$ inch. The different vector quantities are designated according to the symbols shown in figure 62, page 183.

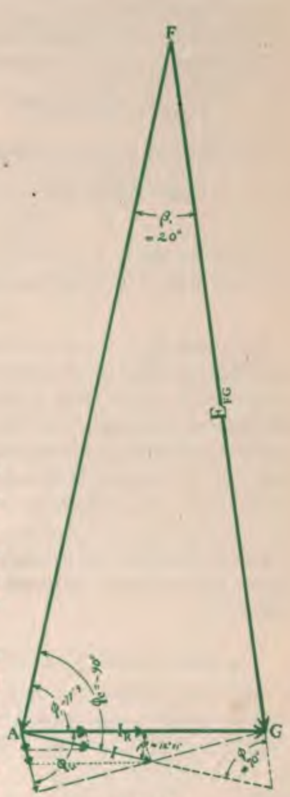

FIG. 64.

The admittance of the non-inductive portion $=0.1$.
The admittance of the inductive portion $=.0256$.
The resultant admittance of the two portions in parallel is $Y=0.1093$.
The admittance of the condenser portion $=.0377$.
The current I_R in the non-inductive branch $=3.858$ amperes.
The current I_L in the inductive branch $=0.989$ ampere.
The resultant current $I = 4.22$ amperes; being the current in the condenser.
The applied pressure $E_{FG}=110$ volts at 60 cycles.
The component pressure $E_{AG}=38.58$ volts; pressure between A and G figure 63.
The component pressure $E_{FA}=111.85$ volts; pressure across condenser.
The ohmic pressure component of the inductive portion is

$$E_{AB}=9.89 \text{ volts.}$$

The reactive pressure component of the inductive portion is

$$E_{BE}=37.28 \text{ volts.}$$

Problem 52c: Construct a diagram to scales given in example 52c, using all values there given except the frequency; change to 25 cycles.

Problem 52d: If a condenser having a capacity of 100 mfs. is connected in parallel with a non-inductive resistance of 10 ohms; the two connected in series with a coil having a resistance of 10 ohms and a coefficient of inductance $L=\frac{1}{10}$ henry, and a pressure of 110 volts at 60 cycles applied to the terminals of the arrangement, find the numerical value of the current in the main line supplying the arrangement, and its phase angle, with reference to the applied pressure.

Figure 65 shows the arrangement of the different portions, and figure 66 shows the vectorial relations of the various currents, pressures and admittances.

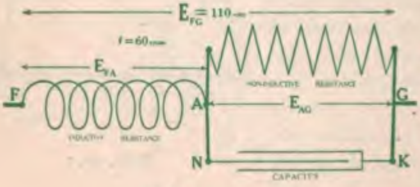

FIG. 65.

GIVEN DATA.

$E=110$ volts.
$f=60$ cycles.
$R=10$ ohms; $R_L=10$ ohms.
$L=\frac{1}{10}$ henry.
$C=100$ mfs. $=.0001$ farad.

Recapitulation: It will be well to compare the currents and the phase angles pertaining to the five different arrangements of the three portions considered.

All in series; current = **4.8** *amperes;* angle of lag = **29° 15′**

All in parallel; current = **11.79** *amperes;* angle of lead = **−6° 47′**.

Series-parallel arrangement; non-inductive resistance outside; current = **1.48** *ampere;* angle of lead = **− 56° 24′**.

Series-parallel arrangement; inductive resistance outside; current = **2.80** *amperes;* angle of lag = **61° 24′**.

Series-parallel arrangement; condenser outside; current = **4.22** *amperes;* angle of lead = **− 70°**.

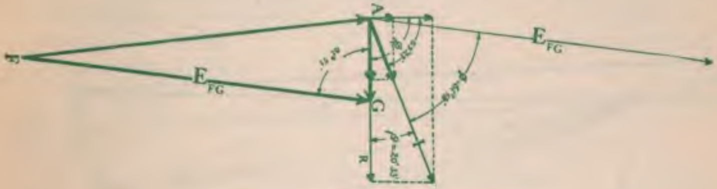

FIG. 66.

Note that the pressure applied to the *arrangement* is in every case 110 volts at 60 cycles.

A *comparative* study of figures 61, 64 and 66, in connection with the corresponding diagrams in figures 60, 63 and 65, will greatly assist in comprehending the fundamental principles involved.

Problem 52e: Construct a vector diagram, (using the same scales given in example 52c) showing the relations of the various pressures, currents and admittances, using all values given in example 52d, except the frequency; change this to 25 cycles.

Combination of Series and Parallel Arrangements.

A large variety of conditions may be imposed in the presentation of problems to be solved, involving *arrangements* of inductive and non-inductive apparatus and condensers. Such conditions might be imposed that it would be difficult if not impossible to solve for the *current*, either the resultant or the component values, without *first* solving for the equivalent admittance of the complete arrangement. To illustrate this statement, the following example is given:

EXAMPLE 53: Find the current, in amperes, in the branch FA of a circuit as shown in figure 67 having values as there designated; the applied pressure E_{FG} between F and G being 110 volts at 25 cycles, in place of 60 ∼ as given in the figure.

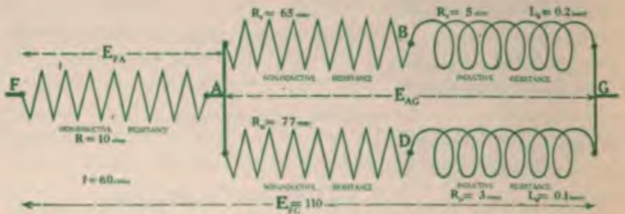

FIG. 67.

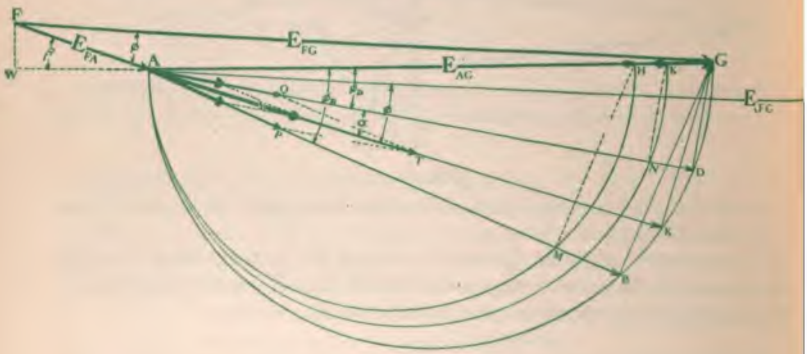

FIG. 68.

GIVEN DATA.

$E_{FG} = 110$ volts; $f = 25$ cycles.
R $= 10$ ohms; non-inductive resistance.
$R_{/} = 65$ ohms; non-inductive resistance.
$R_{//} = 5$ ohms; $L_{//} = 0.2$ henry.
$R_{//} = 77$ ohms; non-inductive resistance.
$R_{iv} = 3$ ohms; $L_{iv} = 0.1$ henry.
$2\pi f = 157$; see page 208.

COMPUTED DATA.

In figure 68.

Vector FG denotes the applied pressure, $E_{FG} = 110$ volts.
Vector AG denotes the pressure $E_{AG} = 88.59$ volts.
Vector FA denotes the pressure $E_{FA} = 22.26$ volts.
Vector AB denotes the ohmic pressure $(R_{,} + R_{,,})\, I_{B} = 80.5$ volts.
Vector AD denotes the ohmic pressure $(R_{,,,} + R_{iv})\, I_{D} = 86.4$ volts.
Vector BG denotes the reactive pressure $2\pi f L_{,,} I_{B} = 36.11$ volts.
Vector DG denotes the reactive pressure $2\pi f L_{iv} I_{D} = 16.95$ volts.

* * *

Vector **AM** $= R_{,,,} + R_{iv} = 80$
Vector **AN** $= R_{,} + R_{,,} = 70$
Vector **HM** $= 2\pi f L_{,,} = 31.4$
Vector **KN** $= 2\pi f + L_{iv} = 15.7$
Vector **AH** $= 76.72$
Vector **AK** $= 81.52$

$\phi_{B} = 24^\circ\, 10'$.
$\phi_{D} = 11^\circ\ \ 7'$.
$\phi = 14^\circ\, 17'$.
$\alpha = 6^\circ\, 43'$.
$\beta = 17^\circ\, 50'$.

* * *

Current vector **AP** $= I_{B} = 1.15$ amperes.
Current vector **AQ** $= I_{D} = 1.08$ amperes.
Current vector **AT** $= I = 2.22$ amperes.

In the vector diagram, of which figure 68 is a reproduction, one inch denoted 10 volts, and impedances were taken same scale; two inches denoted one ampere, while the scale of admittances was 100 inches = 1 unit of admittance.

In this case, under the given conditions, it will be necessary to find the numerical value of the *equivalent* impedance or admittance, in order to find the resulting current.

It will assist greatly in becoming familiar with fundamental principles to treat such a case in the straight-forward and intelligible methods adopted in the treatment of forces in mechanics, and to facilitate the comprehension of this example the steps in the construction of the vector diagram will be explained conjointly with the mathematical process.

Note the symbols applying to different quantities as per figure 62, page 183.

The impedance of branch **ABG** $= Z_{B} = 71.7$ plotted horizontally as **AH**.

The impedance of branch **ADG** $= Z_{D} = 85.9$ plotted horizontally as **AK**.

The pressure component E_{AG} denoted by vector **AG**, figure 68, is common to both branches; but its numerical value is unknown.

The *lag* angle of current $\mathbf{I_B}$ (vector **AQ** figure 68) in branch **ABG** is found from:

$$\tan \phi_{\mathbf{B}} = \frac{2\pi f \mathrm{L}_{,,}}{\mathbf{R}_{,}+\mathbf{R}_{,,}} = \tfrac{31.4}{70} = 0.4485 \text{ from which}$$

$$\phi_{\mathbf{B}} = 11^\circ\ 7'; \text{ plotted below the horizontal.}$$

The *lag* angle of current $\mathbf{I_D}$ (vector **AP** figure 68) in branch **ABG** is found from

$$\tan \phi_{\mathbf{D}} = \frac{2\pi f \mathrm{L}_{\mathrm{iv}}}{\mathbf{R}_{,,,}+\mathbf{R}_{\mathrm{iv}}} = \tfrac{15.7}{80} = 0.4487 \text{ from which}$$

$$\phi_{\mathbf{D}} = 24^\circ\ 10'; \text{ also plotted below the horizontal.}$$

The next step will be to find the numerical values of the admittances of the two branches, and knowing that the angles between the admittances is the same as the angles between corresponding currents, to find the *joint admittance* of the two branches.

The admittance of branch **ABG** $= \dfrac{1}{\mathbf{Z_B}} = \mathbf{Y_B} = .013034$

and is plotted to the scale 100 inches = 1 unit, along the line through **A**, making an angle of 24° 10′ with the horizontal.

The admittance of branch **ADG** $= \dfrac{1}{\mathbf{Z_D}} = \mathbf{Y_D} = .012266$

and is plotted to the scale 100 inches = 1 unit, along the line through **A**, making an angle of 11° 7′ with the horizontal.

The angle between the two admittances is $(24^\circ\ 10' - 11^\circ\ 7') = 13^\circ\ 3'$, and the resultant of the two admittances is found from;

$$\mathrm{Y_{BD}} = \sqrt{.013034^2 + .012266^2 + 2 \times .013034 \times .012266 \times .97417}$$

$$= .025136.$$

$$\mathrm{Y_B} = .013034. \qquad \cos 13^\circ\ 3' = 0.97417$$
$$\mathrm{Y_D} = .012266.$$

The angle α between the admittance of branch **ADG** and the resultant admittance is found from;

$$\cos \alpha = -\frac{\mathrm{Y_B}^2 - \mathrm{Y_D}^2 - \mathrm{Y}^2}{2\mathrm{Y_D}\mathrm{Y}} = +0.99313; \text{ from which}$$

$$\alpha = 6^\circ\ 43'.$$

Through point **A** a line (**AK**) extending to the left of point **A**, of indefinite length is drawn, making an angle of $(\phi_D + \alpha) = 11° \, 7' + 6° \, 43' = 17° \, 50'$, with the horizontal, which will be designated as β. The angle β is an important angle in obtaining certain desirable values.

It is now evident that the *resultant* current; the vectorial resultant of I_B and I_D, multiplied by the resistance of the branch **FA** will give the numerical value of the pressure between **F** and **A**; that the pressure must be denoted along the line **AK** just drawn, and extending to the left of point **A**. Further it is known that if this pressure is denoted by a certain length along **AK**, that the pressure E_{AG} between the points **A** and **G** of the given circuit, must be denoted by a certain length from **A** along the horizontal to the right of **A**, and that the line as **FG** connecting their outer ends must represent 110; the numerical value of the applied pressure.

It will now be in order to establish some relation between the unknown but desirable values, and those values which are known; as follows:

$$E_{FG} = \sqrt{(\overline{WA} + E_{AG})^2 + \overline{WF}^2}$$

$$= \sqrt{(\overline{WA}^2 + 2E_{AG}WA + E_{AG}^2 + \overline{WF}^2}.$$

But $\overline{WA} = E_{FA} \cos \beta$; $\overline{WF} = E_{FA} \sin \beta$, and $E_{FG} = 110$.

Making these substitutions gives: (Page 11.)

$$110 = \sqrt{E_{FA}^2 \cos^2 \beta + 2E_{AG} \, E_{FA} \cos \beta + E_{AG}^2 + E_{FA}^2 \sin^2 \beta}.$$

This may be simplified (by remembering that $\sin^2 \beta + \cos^2 \beta = 1$,) to

$$110 = E_{FA} \sqrt{2E_{AG}E_{FA} \cos \beta + E_{AG}^2 + E_{FA}^2}$$

$$\text{But } E_{FA} = IR = Y_{BD}E_{AG} \, R; \text{ (since } I = YE_{AG})$$

$$\text{Then } 110 = \sqrt{2E_{AG}^2 \, Y_{BD}R \cos \beta + E_{AG}^2 + Y_{BD}^2 \, E_{AG}^2 \, R^2}$$

$$= E_{AG} \sqrt{Y_{BD}^2 \, R^2 + 2Y_{BD}R \cos \beta + 1}$$

Solving the last two equations for the only unknown; E_{AG} gives;

$$E_{AG} = \frac{110}{\sqrt{Y_{BD}^2 R^2 2Y_{BD}R \cos \beta + 1}}$$ being a general equation applicable to any *similar* case, when 110 is replaced by E_{FG}; which might have any numerical value.

The angle denoted by β is equal to 17° 50′.

The numerical values of all quantities expressed in the right hand member of the last equation are known.

$Y_{BD} = .025136$; $R = 10$ ohms $\cos \beta = 0.95195$ $\beta = (\phi_D + \alpha) = 11° \, 7' + 6° \, 43'$.

Solving this equation, gives $E_{AG} = \mathbf{88.59}$ **volts.**

With a scale of 1 inch = 10 volts, this pressure may be laid off on the horizontal from point **A**; giving the vector **AG**.

So far as the diagram is concerned; if compasses are set with a distance of 11 inches between points, and one point placed at point **G** on the vector diagram, while the other compass point is allowed to fall on the line **FA**, the pressure vector E_{FA} is determined.

To compute the numerical value of the pressure E_{FA}, proceed as follows:

$$E_{FA} = Y_{BD}RE_{AC} = .025136 \times 10 \times 88.59 = 22.26 \text{ volts.}$$

Therefore the resultant current **I**, (which is in phase with E_{FA}) $= \frac{22.26}{10} =$ **2.22 amperes. Answer.**

By constructing the semicircles as shown in figure 68, page 190, the value of an *equivalent reactive pressure* is denoted by **GK**; of an *equivalent* ohmic pressure by **AK**, and of the individual branch pressure components.

This means that a single resistance and a single reactance each having the values designated, might replace the several other resistances and reactances, permitting the same resultant current to exist, with the same applied pressure and frequency.

It may be noted that the product of the numerical values of I_B and **AH**, gives the numerical value of the pressure line **AG**, and that the product of I_D and **AK**, gives the *same value*; since the pressure E_{AG} is common to both branches.

The smaller current times the larger impedance is equal to the larger current times the smaller impedance.

PROBLEM 53: Given an arrangement consisting of inductive and non-inductive resistances, connected together as indicated in figure 67, page 190, having values there stated. Compute the numerical value and the phase relation, relative to the applied pressure E_{FG}, *of the resultant or main-line current.*

Problem 53a: Given an arrangement of inductive and non-inductive resistances as indicated in figure 67, page 190, with numerical values as there stated, except the frequency. Change the frequency to 15 cycles and find the numerical value, and the phase relation of the resultant or main-line current, relative to the applied pressure E_{FG}.

Draw a vector diagram, as indicated in figure 68, page 190, using scales there given, showing the vector relations of all pressure and current components, and all impedances and admittances.

EXAMPLE 54: Given an arrangement consisting of inductive and non-inductive resistances, and a condenser arranged as shown in figure 69, with numerical values as there indicated, find the numerical value, in amperes, of the current I in the branch FA, and the phase relation of this current with the applied pressure E_{FG}.

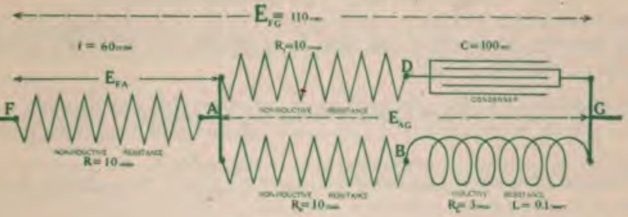

FIG. 69.

GIVEN DATA.

E_{FG} =110 volts.
f =60 cycles.
R =10 ohms; $R_{,}$ =10 ohms,
$R_{,,}$ =10 ohms; R_L =3 ohms.
$L = \frac{1}{10}$ henry; $C = 100$ mfs. $= \frac{1}{10,000}$ farad.

It is not possible in this case to find at once the value and the phase relations of the branch currents, since the pressure component between **A** and **G** is not known. It will therefore be necessary either to find the equivalent impedance or admittance of the complete circuit first; from which the value of the current in **FA** as well as the branch currents may be computed; or to find the numerical value of the pressure between **AG**, and of the pressure between **FA**, and from these find the value of the current in the non-inductive branch **FA**.

Let I_B denote the current in the branch **ABG**.

Let I_D denote the current in the branch **ADG**.

Let **I** denote the current in the portion **FA**; (main line current.)

The impedance of the branch **ABG** is expressed by:

$$Z_B = \sqrt{(R_{,}+R_L)^2+(2\pi f L)^2},$$

and by substituting in this expression the proper numerical values, gives;

$\sqrt{169 + 1421.29}$; from which the value 39.87 is obtained.

The current $I_B = \frac{E_{AG}}{39.87}$

The angle of lag of the current $\mathbf{I_B}$, in the branch **ABG**, behind the *pressure applied to the branch*, (denoted by $\mathbf{E_{AG}}$) may be denoted by $\phi_\mathbf{B}$ and is found from:

$$\tan\ \phi_\mathbf{B} = \frac{2\pi f \mathrm{L}}{\mathrm{R}_{,} + \mathrm{R_L}} = \tfrac{37.7}{13} = 2.9, \text{ from which}$$

$$\phi_\mathbf{B} = 70^\circ\ 59'. \quad \text{The } \cos\ \phi_\mathbf{B} = 0.32557$$

$\phi_\mathbf{B}$ denotes an angle of lag, and $\mathbf{I_B}$ is behind the pressure component $\mathbf{E_{AG}}$ at an angle of 70° 59′.

The admittance of this branch is; $\mathbf{Y_B} = \dfrac{1}{\mathbf{Z}} = \tfrac{1}{39.87} = .025332.$

The conductance of this branch is; $\mathrm{g_B} = \dfrac{\mathrm{R}_{,} + \mathrm{R_L}}{\mathbf{Z_B}^2} = \tfrac{13}{1590.28} = .008174.$

The susceptance of this branch is; $\mathrm{b_B} = \dfrac{\mathbf{X_B}}{\mathrm{Z_B}^2} = \tfrac{37.17}{1590.29} = .023700.$

The impedance triangle applying to the branch **ABG** is shown by the triangle **AMJ** in figure 70, page 197.

$$\mathrm{g_B} = \frac{\cos\ \phi_\mathbf{B}}{\mathrm{Z_B}} = \mathrm{Y_B} \cos\ \phi_\mathbf{B}.$$

$$\cos\ \phi_\mathbf{B} = \frac{\mathrm{R}_{,} + \mathrm{R_L}}{\mathrm{Z_B}}.$$

The impedance of the branch **ADG** is expressed by:

$\mathbf{Z_D} = \sqrt{\mathbf{R}_{,,}{}^2 + \left(-\dfrac{1}{2\pi f \mathrm{C}}\right)^2}$, and by substituting the proper numerical values, $(\sqrt{100 + 26.5^2})$, the value 28.34 is obtained. The square of the reciprocal of $\dfrac{1}{2\pi f}$ is given on page 109.

The current $\mathrm{I_D} = \dfrac{\mathrm{E_{AG}}}{28.34}$.

The angle $\phi_\mathbf{D}$ between the current $\mathrm{I_D}$ and the pressure $\mathrm{E_{AG}}$ is obtained from:

$$\tan\ \phi_\mathbf{D} = -\frac{1}{2\pi f \mathrm{CR}_{,,}} = -\frac{1}{377 \times \frac{100}{1000000} \times 10} = -\tfrac{1000}{377} = -2.65.$$

Therefore $\phi_\mathbf{D} = -69^\circ\ 20'$ (to nearest minute) which is an angle of *lead*. $\mathbf{I_D}$ is ahead of $\mathbf{E_{AG}}$ in phase relation.

The admittance of this branch is; $\mathbf{Y_D} = \frac{1}{\mathbf{Z_D}} = \frac{1}{28.34} = .03528.$

The conductance of this branch is; $\mathrm{g_D} = \frac{\mathrm{R}_{\prime\prime}}{\mathbf{Z_D}^2} = \frac{10}{803.1556} = .012444.$

The susceptance of this branch is; $\mathrm{b_D} = \frac{\mathbf{X_D}}{\mathbf{Z}^2} = \frac{26.52}{803.1556} = .03301.$

The foregoing results have been plotted to scale in figure 70 which is a diagram of impedances, resistances, reactances, admittances, pressures and currents.

For convenience the pressure (E_{AG}) between **A** and **G** is taken as horizontal datum, and denoted by the line **AG**, as is done in preceding examples.

If it is desired that the resultant or main line current be employed as the horizontal datum the figure may be so held that the vector **FA** is horizontal; if the applied pressure is desired as horizontal datum turn the figure so that $\mathbf{E_{FG}}$ is horizontal. The same procedure may be applied to the vector diagrams representing other examples.

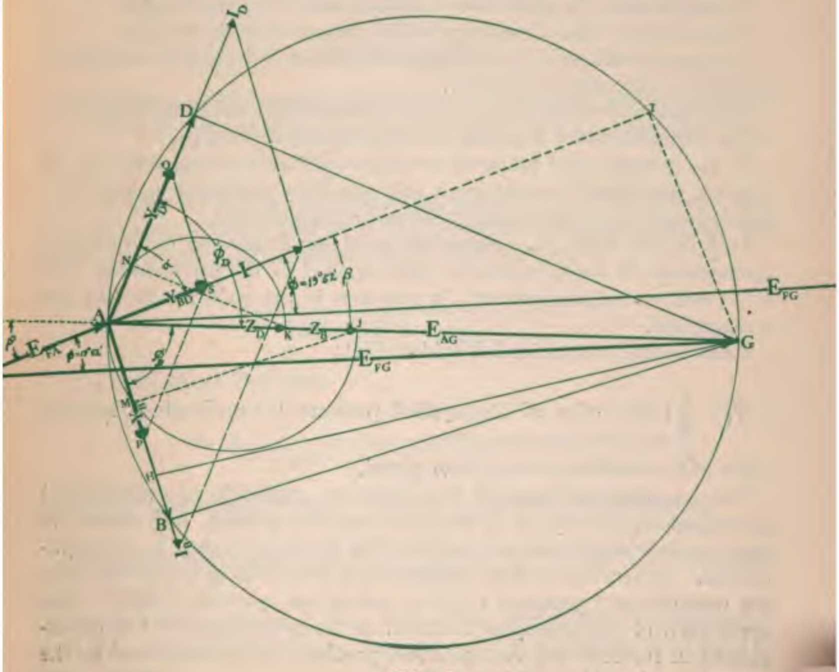

FIG. 70.

The current, I_B in the branch **ABG** will be in phase with the line AB denoting the ohmic pressure component of E_{AG}, which lags (E_{AG}) **AG** by an angle of 70° 59′.

In the vector diagram, figure 70, **AM** $= R_{,} + R_L = 10 + 3 = 13$ ohms.

The reactive-pressure-component of this branch is denoted by the line **BG**.

The impedance of the branch **ABG** is denoted by the line **AJ**.

This impedance multiplied by the current I_B will give the component E_{AG} of the applied pressure. The component E_{AG} being denoted by the line **AG**.

Likewise in the same vector diagram, **AN** $= R_{,,} = 10$ ohms;

The reactive pressure-component of the branch **ADG** is denoted by the line **GD**.

The impedance of the branch **ADG** is denoted by the line **AK**.

This impedance multiplied by the current I_D will also give the component E_{AG} (denoted by the line **AG**) of the applied pressure, E_{FG}.

The admittance Y_B of the branch **ABG** is denoted by the line **AP**.

The admittance Y_D of the branch **ADG** is denoted by the line **AQ**.

The joint admittance Y_{BD} of the two branches is denoted in magnitude and phase relation, by the line **AS**.

The *resultant* current (of I_B and I_D) is numerically equal to the product of the joint admittance Y_{BD} and the component pressure E_{AG}.

In the examples and problems treating of different arrangements of inductive, non-inductive resistances and capacities, explained in preceding pages, the *total* admittance may be found from the relation.

$I = E\ Y_t$; in which Y_t denotes the total admittance of the complete arrangement, **E** the pressure, in volts, applied to the arrangement, and **I** the total or resultant current, in amperes, in the main line feeding the arrangement.

From the above equation it is evident that;

$Y_t = \frac{I}{E}$; the value of the applied pressure is usually given, and the value of the resultant current is computed.

The foregoing fundamental principles are applicable in experimental investigations, as well as in practical engineering work, and should be very carefully considered and studied with reference to their useful applications. In very many cases the discussion presented in connection with the examples and problems helps to answer the question "*why?*" The application of the principles discussed in the preceding text will be explained in the following examples and problems in this book and in the examples and problems cf **Volume II.**

PROBLEM 54: Given an arrangement consisting of inductive and non-inductive resistances, and a condenser, arranged as shown in figure 69, page 195 with values there given except the frequency which should be changed to 25 cycles. With this change, find the numerical value, in amperes, of the current I, in the branch FA, and its phase relation with the applied pressure, denoted by $\mathbf{E}_{\mathrm{FG}}$. *Plot a diagram, as in figure 70, using scales as there stated, showing the numerical values, and phase relations, of all pressure and current components.*

Problem 54a: Given an arrangement consisting of inductive and non-inductive resistances, and a condenser, arranged as shown in figure 69, with values there given except the frequency, which should be changed to 15 cycles. With this change, find the numerical value, in amperes, of the resultant or main line current, and plot a vector diagram, using the scales given for figure 70, showing the numerical values and the phase relations of the various pressure and current components.

RESONANCE.

Whenever a circuit contains capacity and inductance, either connected in series or in parallel, there is always the possibility of producing "*resonance*".

If the different portions designated are connected together in series, and an alternating-pressure applied to the terminals of the arrangement, the expression for the alternating-current in the circuit is;

$$\mathbf{I} = \frac{\mathbf{E}}{\sqrt{\mathbf{R}^2 + (2\pi f \mathbf{L} - \frac{1}{2\pi f \mathbf{C}})^2}} \cdot$$

It is to be noted that when

$$2\pi f \mathbf{L} = \frac{1}{2\pi f \mathbf{C}} \text{ in the preceding equation, then } \mathbf{I} = \frac{\mathbf{E}}{\mathbf{R}}$$

which is the expression for ohm's law as in direct-current engineering.

Considering the expression $2\pi f\mathbf{L} - \frac{1}{2\pi f\mathbf{C}}$, it may be seen that if f is kept constant, while **L** is varied, then **C** must be varied *inversely* compared with the variation in **L** in order that the value of the whole expression remain constant.

On the other hand assuming given values for **L** and **C**; as 1.01 henry and 10 microfarads, giving;

$$\tfrac{101}{100}\pi f \cdot - \frac{1}{2\pi f \tfrac{10}{1000000}}; \text{ or } \tfrac{101}{100}\pi f - \frac{100000}{2\pi f}$$

then as the frequency decreases, the less the value of the term $\tfrac{101}{100}\pi f$ becomes as compared with the term $\frac{100000}{2\pi f}$; or very *high* frequencies cause the numerical value of the term $\frac{1}{2\pi f}$ to become *small* compared with the numerical value of $2\pi f$, in the general equation, page 199.

Somewhere between the two extreme values of these terms there must be some value of f that will cause the numerical value of $2\pi f\mathbf{L}$ to become equal to the numerical value of $\frac{1}{2\pi f\mathbf{C}}$, for any given values of **L** and **C**. For example; in the given case, where $\mathbf{L} = 1.01$ henry, and $\mathbf{C} = \tfrac{1}{100000}$ farad, in order that $2\pi f\mathbf{L}$ shall equal $\frac{1}{2\pi f\mathbf{C}}$ what must be the value of **C**?

In this case $2\pi f = \frac{10{,}000{,}000}{2\pi f \times 101}$ or $f^2 = \frac{10{,}000{,}000}{4\pi^2 \times 101} = \frac{100{,}000}{4\pi^2 \times 1.01}$

From which $f = \frac{\sqrt{100{,}000}}{2\pi \times \sqrt{1.01}} = \frac{1}{2\pi}\ \frac{\sqrt{100{,}000}}{\sqrt{1.01}} = 50$ cycles.

Again considering the expression; $2\pi f\mathbf{L} - \frac{1}{2\pi f\mathbf{C}}$ assuming given fixed values for **L** and **C** it may be proved by substituting numerical values for f that when the value of f is *greater* than 50 cycles, the value of $2\pi f\mathbf{L}$ is greater than the value of $\frac{1}{2\pi f\mathbf{C}}$; while a value of f *less* than 50 cycles will make the value of $2\pi f\mathbf{L}$ less than the value of $\frac{1}{2\pi f\mathbf{C}}$.

Whenever the values of the various quantities are such that the numerical value of $2\pi f\mathbf{L} = \frac{1}{2\pi f\mathbf{C}}$, then the condition of **resonance** is said to obtain in any circuit.

It is necessary to give careful consideration to the matter of resonance in calculations pertaining to proposed new transmission lines. If after adopting a certain size of wire and given spacing of the wires, the calculations of the capacity and inductance (**L**) show that the value of $2\pi f L$ for the given line* is equal to $\frac{1}{2\pi f C}$ ** of the same line, f being the proposed frequency of the applied or generator pressure, then the spacing of the wires may be *changed* in order that the relative values of **L** and **C** may be changed, so that the condition of resonance no longer obtains. If the conditions for resonance obtain when an inductive portion of a circuit is in *series* with a condensive portion, than very great *pressures* (pressure component) are maintained between the terminals of the inductive portion, and the terminals of the condensive portion, which tends to break down or puncture the insulation of the conductors. If resonance obtains when an inductive portion is in *parallel* with a capacity then a very large current exists in the circuit, tending to melt the conductors.

It is evident that at least two undesirable features attend the condition of resonance; excessive pressure or excessive current.

One desirable feature of resonance is unity power factor. Power factor is discussed on page 127.

When the power factor of a circuit is unity, the cosine of the angle between the pressure and current must be 1; or the angle of lag (or lead) must be 0°.

When $2\pi f L = \frac{1}{2\pi f C}$, then $\tan \phi = \frac{2\pi f L - \frac{1}{2\pi f C}}{R}$ becomes equal to *zero*.

In other words since the tangent of the angle of lag (or lead) is zero, the angle of lag (or lead) is zero, and its *cosine* becomes *unity*.

---

*See page 121. **See Vol. II.

EXAMPLE 56: Find the necessary inductance, in henrys, to produce resonance in a circuit containing a resistance of 10 ohms, in series with a condenser having a capacity of $2\frac{6}{10}$ microfarads, if the frequency of the applied pressure is 100 cycles per second.

DATA.

$f = 100$ cycles per second.
$f^2 = 10000$
$C = .0000026$ farad.
$4\pi^2 = 39.4784$.

The condition for complete resonance is expressed by:

$$2\pi f L = \frac{1}{2\pi f C}.$$

in which **L** denotes the coefficient of inductance of the coil, expressed in henrys, and C the capacity of the condenser expressed in *farads*, while f denotes the frequency of the applied pressure in cycles per second.

Solving for **L** gives;

$$L = \frac{1}{4\pi^2 f^2 C}.$$ After substituting numerical values gives

$$L = \frac{1}{394784 \times .0000026} = 0.975 \text{ henry.}$$ **Answer.**

PROBLEM 56: Find the inductance, in henrys, necessary to produce resonance in the circuit mentioned in example 56 if the frequency is changed to 60 cycles per second. **0.3518 henrys. Answer.**

Consult table, page 208.

Problem 56a: Find the inductance in henrys necessary to produce resonance in a circuit consisting of a resistance of $\frac{1}{1000}$ ohm, connected in series with a condenser having a capacity of $\frac{1}{1000}$ microfarad, if the frequency of the applied pressure is 20,000 cycles per second.

Problem 56b: Find the inductance in henrys necessary to produce resonance in the circuit mentioned in problem 56a if the frequency is changed to 200,000 cycles per second.

Problem 56c: Compute the current in amperes in a circuit consisting of a coil having a resistance $\mathbf{R} = 1$ ohm and a coefficient of inductance $L = 2.026$ henrys, if connected in series with a condenser having a capacity $C = 20$ microfarads, when an A. C. pressure of 110 volts at 25 cycles per second is applied to the arrangement.

Problem 56d: Draw a curve, to scale, on section paper, showing the variation of current in a circuit consisting of a coil having a resistance **R** of 5 ohms, and a variable inductance **L** (**L** being made variable by moving an iron core in or out) connected in series with a condenser having a fixed capacity C of 20 microfarads, when a constant A. C. pressure of 110 volts at a frequency $f = 25 \sim$ is applied to the terminals of the arrangement.

Compute at least ten values for the current, one value being the *maximum* value.

Assume values for **L** from ½ to 2½ henrys.

Plot values of **L** along the horizontal datum axis as abscissae and corresponding values of current, in amperes, vertically as ordinates.

EXAMPLE 57: Find the pressure, in volts, across the terminals of the condenser mentioned in EXAMPLE 56, page 204, **when resonance occurs, if the applied pressure is 100 volts.**

Find the phase relation of the computed pressure with the applied pressure.

Find the phase relation of the computed pressure with the pressure across the inductive portion of the circuit.

The same current exists in the condensive portion of the circuit as in the inductive portion, since it is a "series" circuit.

The current is; $\mathbf{I}=\frac{100}{10}=10$ amperes.

The current in the inductive portion multiplied by the inductive reactance, will give the reactive pressure of this portion.

The reactance of this portion $=\mathbf{X}_{\mathrm{L}}=612.61$.

The reactive pressure of the inductive portion $=10\times612.61=$ **6126.1 volts.**

The pressure between the terminals of the condenser is

$$\sqrt{100^2+6126.9^2} \quad = \quad \textbf{6126.9 volts.} \quad \textbf{Answer.}$$

The pressure component of the applied pressure, across the inductive portion, is very nearly equal to the reactive pressure of this portion.

The angle of lag ϕ_{L}, between the main line current **I**, and the pressure just obtained is found from;

$$\tan\phi_{\mathrm{L}} = \frac{2\pi f\mathrm{L}}{\mathrm{R}} = \frac{612.61}{10} = 61.261 \text{ from which}$$

$$\boldsymbol{\phi_{\mathrm{L}} = 89^\circ\ 4'.}$$

The numerical value of the pressure component between the terminals of the condenser, may be found by multiplying the capacity reactance, $\frac{1}{2\pi f\mathrm{C}}$ by the current of 10 amperes; giving very nearly the same value as for the inductive portion. The angle of lead ϕ_{C} of the condenser pressure component in advance of the applied pressure of 100 volts is found from;

$$\tan\phi_{\mathrm{C}} = -\frac{1}{2\pi f\mathrm{CR}} = -61.27.$$

ϕ_{C} is therefore very nearly $-89^\circ\ 4'$. **(The minus sign denoting an angle of *lead*.)**

The angle between the two component pressures is

$$89° \; 4' + 89° \; 4' = \mathbf{178° \; 8'} \text{ very nearly.}$$

If $\mathbf{E_L}$ denotes the inductive component while **EC** denotes the condenser component, the following relation is true;

$$\mathrm{E} = 100 = \sqrt{\mathrm{E_L}^2 + \mathrm{E_C}^2 + 2\mathrm{E_L}\,\mathrm{E_C} \cos 178° \, 8'} \; ;$$

by which the values may be checked with each other.

PROBLEM 57: *Compute the "resonance" frequency of an alternating-pressure, such that when applied to the terminals of an arrangement consisting of a coil having a coefficient of inductance* L=0.72 *henry, and a resistance of* 5 *ohms, connected in series with a condenser having a capacity of* $\frac{1}{100000}$ *farad, ohms law will obtain.* **60 cycles. Answer.**

Problem 57a: Compute the necessary resonance frequency required to realize the conditions of ohms law, in *PROBLEM* 57, provided the value of the capacity is reduced to $\frac{1}{1000000}$ microfarad; all other values as stated.

TABLE I.

The value of π is taken as 3.14159. $\frac{1}{2\pi} = 0.15915$. $\frac{1}{4\pi^2} = .0252\dot{7}$.

Values of Frequency, f, in Cycles per Second.	Corresponding Values of $2\pi f$	Corresponding Values of $(2\pi f)^2 = 4\pi^2 f^2$	Corresponding Values of $\frac{1}{2\pi f}$	Corresponding Values of $\frac{1}{(2\pi f)^2}$
1	6.28318	39.438	.1591	.02533
2	12.56636	157.753	.0795	.006332
3	18.84954	354.945	.0535	.002814
4	25.13272	631.576	.0397	.001583
5	31.41590	986.588	.0318	.0010857
6	37.69908	1421.290	.0265	.0007036
7	43.98226	1934.240	.0227	.00051694
8	50.26544	2526.067	.0198	.00039579
9	56.54871	3197.902	.0176	.0003127
10	62.83189	3947.835	.01591	.0002533
11	69.11498	4776.192	.0144	.0002093
12	75.39816	5685.160	.0132	.0001759
13	81.68134	6671.622	.0122	.00014988
14	87.96452	7736.961	.0113	.00012924
15	**94.24779**	**8882.6289**	**.01061**	**.0001259**
16	100.53088	10106.290	.0099	.000098946
17	106.81406	11408.376	.0093	.000087648
18	113.09724	12791.610	.0088	.00007818
19	119.38042	14251.584	.0083	.000070167
20	125.66360	15791.340	.0078	.00006332
21	131.94678	17410.802	.0075	.000057439
22	138.22996	19107.532	.0072	.000052335
23	144.51314	20883.140	.0069	.00004789
24	150.79632	22740.640	.0066	.000043977
25	**157.07950**	**24673.937**	**.006366**	**.000040529**
26	163.36268	26687.365	.0061	.00003747
27	169.64586	28779.719	.0060	.000034747
28	175.92904	30951.027	.0057	.000032509
29	182.21222	33201.293	.0055	.000030119
30	188.49540	35530.515	.005305	.00002815
31	194.71858	37938.695	.0051	.000026358
32	201.06176	40425.831	.0049	.000024737
33	207.34494	42991.924	.0048	.00002326
34	213.62812	45636.973	.0047	.00002191
35	219.91130	48360.979	.0045	.000020678
36	226.19448	51163.942	.0044	.00001910
37	232.47766	54045.862	.0043	.00001850
38	238.76084	51006.738	.0042	.00001754
39	245.04402	60046.521	.0041	.00001665
40	251.32720	63165.361	.0040	.0001583
41	257.61038	66363.107	.0039	.000015068
42	263.89356	69639.311	.0038	.000014326
43	270.17674	72995.470	.0037	.000018774
44	276.45992	76430.087	.0036	.000013084
45	282.743	79943.610	.0035	.00001265
46	289.026	83536.190	.0035	.00001197
47	295.309	87207.404	.0034	.000011467
48	301.592	90958.120	.0033	.000010984
49	307.875	94787.520	.0032	.00001050
50	**314.159**	**98695.877**	**.003183**	**.0000101321**

[Table I. Continued.]

Values of Frequency, f, in Cycles per Second.	Corresponding Values of $2\pi f$	Corresponding Values of $(2\pi f)^2 = 4\pi^2 f^2$	Corresponding Values of $\frac{1}{2\pi f}$	Corresponding Values of $\frac{1}{(2\pi f)^2}$
51	320.442	102683.075	.00312	.00000973869
52	326.725	106749.225	.00306	.00000936772
53	333.008	110894.994	.00300	.00000901756
54	339.291	115119.060	.00294	.00000868667
55	345.579	119422.080	.00290	.0000083736
56	351.858	123804.052	.00284	.0000080772
57	358.141	128264.259	.00297	.0000077963
58	364.424	132801.93	.00275	.0000075298
59	370.707	137418.49	.00270	.0000072767
60	**376.990**	**142122.06**	**.002652**	**.0000070362**
61	383.274	146895.89	.00261	.0000068074
62	289.557	151756.99	.00257	.0000065895
63	295.840	156689.30	.00253	.0000063820
64	402.122	161700.49	.00249	.0000061846
65	408.406	166798.72	.00245	.0000059953
66	414.689	171967.79	.00241	.0000058150
67	420.973	177215.74	.00238	.0000056427
68	427.256	182551.10	.00234	.0000054780
69	433.539	187956.93	.00230	.0000053203
70	439.822	193441.63	.00228	.0000051694
71	446.105	199005.21	.00225	.0000050248
72	452.389	204556.71	.00221	.0000048862
73	458.672	210378.17	.00218	.0000047533
74	464.955	216187.80	.00215	.0000046257
75	471.238	222067.13	.00212	.0000045031
76	477.521	227915.3	.00209	.0000043854
77	483.804	234062.5	.00206	.0000042722
78	490.087	240267.8	.00204	.0000041634
79	496.371	246382.0	.00202	.0000040586
80	502.654	252657.0	.00200	.0000039578
81	508.937	259019.2	.00196	.0000038607
82	515.220	265451.4	.00194	.0000037671
83	521.503	271962.3	.00192	.0000036769
84	527.786	278557.8	.00189	.0000035899
85	534.069	285232.6	.00187	.0000035059
86	540.352	291978.4	.00185	.0000034248
87	546.635	298808.1	.00183	.0000033465
88	552.918	305727.4	.00181	.0000032709
89	559.201	312704.8	.00179	.0000031978
90	565.484	319776.3	.00177	.0000031272
91	571.768	326910.0	.00173	.0000030588
92	578.051	334141.8	.00173	.0000029927
93	584.334	341443.5	.00171	.0000029287
94	590.617	348828.0	.00169	.0000028667
95	596.900	356293.9	.00167	.0000028066
96	603.184	363832.0	.00165	.0000027485
97	609.467	371450.1	.00164	.0000026921
98	615.750	379146.0	.00163	.0000026374
99	622.034	386924.0	.00161	.0000025844
100	**628.318**	**394783.50**	**.0015915**	**.0000025330**
101	634.601	402718.6	.00158	.0000024831
102	640.884	410732.3	.00156	.0000024346
103	647.168	418826.4	.001545	.0000023876
104	653.451	426998.2	.00153	.0000023419
105	659.734	435248.9	.001515	.0000022975

[Table I. Continued.]

Values of Frequency, f, in Cycles per Second.	Corresponding Values of $2\pi f$	Corresponding Values of $(2\pi f)^2 = 4\pi^2 f^2$	Corresponding Values of $\frac{1}{2\pi f}$	Corresponding Values of $\frac{1}{(2\pi f)^2}$
106	666.017	443578.6	.00150	.0000022543
107	672.300	450988.3	.001488	.0000022124
108	678.583	460484.8	.001473	.0000021716
109	684.67	469051.8	.001460	.0000021320
110	691.150	477688.3	.001447	.0000020934
111	697.433	486412.8	.001433	.0000020558
112	703.716	495216.2z	.001420	.0000020193
113	709.999	504098.6	.001408	.0000019837
114	716.283	513061.3	.001395	.0000019490
115	722.565	522101.2	.001383	.0000019153
116	728.848	531220.7	.001371	.0000018824
117	735.132	590419.1	.001360	.0000018504
118	741.415	549686.5	.001349	.0000018191
119	747.698	559052.9	.001338	.0000017887
120	753.981	56848.2	.001326	.0000017590
121	760.264	577995.2	.001315	.0000017300
122	766.548	587583.5	.001304	.0000017018
123	772.831	597266.2	.001293	.0000016742
124	779.114	607012.4	.001283	.0000016473
125	**785.397**	**616847.4**	**.001272**	**.0000016211**
126	791.680	626630.0	.00126	.0000015955
127	797.763	636745.4	.00125	.0000015704
128	804.247	646736.4	.00124	.0000015460
129	810.530	656910.2	.00123	.0000015221
130	816.813	667162.2	.00122	.0000014988
131	823.099	677491.9	.001216	.0000014760
132	829.382	687874.5	.001208	.0000014537
133	835.665	698335.99	.001198	.0000014319
134	841.948	708862.96	.001189	.0000014106
135	848.232	719497.52	.001179	.0000013898
136	854.515	730204.40	.00117	.0000013695
137	860.798	740973.19	.00116	.0000013495
138	867.081	751827.72	.00115	.0000013300
139	873.364	762764.67	.001146	.0000013101
140	879.648	773766.52	.001138	.0000012923
141	855.931		.001129	.0000012740
142	892.214	796020.84	.00112	.0000012562
143	898.497		.00111	.0000012387
144	904.780	818226.84	.001105	.0000012215
145	311.064		.001097	.0000012194
146	917.347	841512.68	.00109	.0000011883
147	923.630		.00108	.0000011722
148	929.913	864751.20	.001075	.0000011564
149	936.196		.001068	.0000011409
150	942.477	888268.52	.00106	.0000011257
1000	6283.189	39478350.9124	.0001591	.000253
2000	12566.36	157913403.	.0000795	.0000633
150000	942477.	888262895529.0	.000001061	.0000000000112

TABLE II.

Table of the **Reciprocals of Capacities** from 0.11 to 18 *microfarads*, or from .00000011 .000018 *farads*.

If the reciprocal of capacities, expressed as farads is desired, **multiply** the tabular values here ren by 1,000,000 (one million).

A	B	C	D	E	F	G	H	I	J
.01	.02	.03	.04	.05	.06	.07	.08	.09	0.1
9.090	8.333	7.692	7.142	6.666	6.250	5.882	5.555	5.263	5.000
4.761	4.545	4.347	4.166	4.000	3.846	3.703	3.571	3.448	3.333
3.225	3.125	3.030	2.941	2.857	2.777	2.702	2.631	2.564	2.500
2.439	2.381	2.325	2.272	2.222	2.174	2.127	2.083	2.041	2.000
1.961	1.923	1.886	1.851	1.818	1.785	1.754	1.724	1.695	1.666
1.639	1.612	1.587	1.562	1.538	1.515	1.492	1.470	1.449	1.428
1.408	1.388	1.369	1.351	1.333	1.316	1.298	1.282	1.265	1.250
1.234	1.219	1.204	1.190	1.176	1.162	1.149	1.136	1.123	1.111
1.098	1.087	1.075	1.063	1.052	1.041	1.031	1.020	1.010	1.000
.9900	.9804	.9708	.9615	.9523	.9434	.9345	.9259	.9174	.9090
.4975	.4950	.4926	.4902	.4878	.4854	.4830	.4807	.4784	.4761
.3322	.3311	.3300	.3289	.3278	.3267	.3257	.3246	.3236	.3225
.2493	.2487	.2481	.2475	.2469	.2463	.2457	.2451	.1447	.2439
.1996	.1992	.1988	.1984	.1980	.1976	.1972	.1968	.1964	.1961
.1663	.1661	.1658	.1655	.1652	.1650	.1647	.1644	.1642	.1639
.1426	.1424	.1422	.1420	.1418	.1416	.1414	.1412	.1410	.1408
.1248	.1246	.1245	.1243	.1242	.1240	.1239	.1237	.1236	.1234
.1109	.1108	.1107	.1106	.1105	.1103	.1102	.1101	.1100	.1098

K	L	M	N	O	P	Q	R	S
1	2	3	4	5	6	7	8	9
.9900	.4761	.3225	.2439	.1961	.1639	.1408	.1234	.1098
.8333	.4545	.3125	.2381	.1923	.1612	.1388	.1219	.1087
.7692	.4347	.3030	.2325	.1886	.1587	.1369	.1204	.1075
.7142	.4166	.2941	.2272	.1851	.1562	.1351	.1190	.1063
.6666	.4000	.2857	.2222	.1818	.1538	.1333	.1176	.1052
.6250	.3846	.2777	.2174	.1785	.1515	.1316	.1162	.1041
.5882	.3703	.2702	.2127	.1754	.1492	.1298	.1149	.1031
.5555	.3571	.2631	.2083	.1724	.1470	.1282	.1136	.1020
.5263	.3448	.2564	.2041	.1695	.1449	.1265	.1123	.1010
.5000	.3333	.2500	.2000	.1666	.1428	.1250	.1111	.1000
.3333	.2500	.2000	.1666	.1428	.1250	.1111	.1000	.0909
.2500	.2000	.1666	.1428	.1250	.1111	.1000	.0909	.0833
.2000	.1666	.1428	.1250	.1111	.1000	.0909	.0833	.0769
.1666	.1428	.1250	.1111	.1000	.0909	.0833	.0769	.0714
.1428	.1250	.1111	.1000	.0909	.0833	.0769	.0714	.0666
.1250	.1111	.1000	.0909	.0833	.0769	.0714	.0666	.0625
.1111	.1000	.0909	.0833	.0769	.0714	.0666	.0625	.0588
.1000	.0909	.8033	.0769	.0714	.0666	.0625	.0588	.0555

The values of capacities, expressed in *farads*, necessary to be connected in circuit with a coil having a variable inductance, varying in steps of unit value, from 1 to 50 henrys, have been computed from the equation:

$$2\pi f \mathrm{L} = \frac{1}{2\pi f \mathrm{C}} \text{ ; from which,}$$

$$\mathrm{C} = \frac{1}{4\pi^2 f^2} \times \frac{1}{\mathrm{L}} = \frac{1}{39.478416} \times \frac{1}{f^2} \times \frac{1}{\mathrm{L}}, \text{ and}$$

arranged in table III, on page 213.

If the frequency f is kept at a constant value, and **L** is varied, then **C** varies *inversely* with **L**, and inversely as the *square* of the frequency.

If for example;

$$f = 15 \sim, \text{ then } \frac{1}{4\pi^2 f^2} = .000112579. \quad (\pi = 3.14159)$$

If $f = 60 \sim$, then $\frac{1}{4\pi^2 f^2} = .0000070362$ which is $\frac{1}{16}$ of the value when $f = 15 \sim$. $(15 \times 4 = 60)$

The values have been tabulated on page 213 for the common frequencies of 15, 25, 50 and 60 cycles, for ready reference; but values for other frequencies may be readily found from the values tabulated.

Suppose the value of capacity, in *farads*, is desired that will neutralize an inductance of 5 henrys and produce resonance in a circuit in which the frequency of supply is 30 $\sim$. Look in column headed 60 $\sim$ and multiply the value corresponding to 5 henrys, by 4; or look in column headed 15 $\sim$; *divide* the value corresponding with 5, by 4.

The value of capacity, necessary to produce resonance, when connected with 10 henrys, if the frequency is 100 $\sim$, is .000001013 divided by 4.

The value of capacity, in farads, necessary to produce resonance, when connected with 1 henry, in a circuit in which the frequency is 50,000, is found from column headed 50 $\sim$ by

dividing .00001013 by $(1000)^2 = 1,000,000 = \overline{10}^6$; which is

$\frac{.00001013}{1000000} = .00000000001013$ farad. $(= .00001013$ microfarad)

If values are desired in *microfarads* instead of in farads, divide the tabular values by 1,000,000 (one million) or 10^6.

The foregoing explanations apply also to tables on pages 214 and 215.

TABLE III.

The following table gives the values of capacity, expressed in *farads* necessary to neutralize inductances from 1 to 50 henrys, and produce complete resonance for circuits in which the frequency of the applied pressure is as given at the head of each column.

$\pi = 3.14159; \quad \pi^2 = 9.8695877; \quad 4\pi^2 = 39.478416.$

Frequency = **15**∼		Frequency = **25**∼		Frequency = **50**∼		Frequency = **60**∼	
Henrys	Farads	Henrys	Farads	Henrys	Farads	Henrys	Farads
1	.00011257	**1**	.00004052	**1**	.00001013	**1**	.000007036
2	.00005628	**2**	.00002026	**2**	.000005066	**2**	.000003518
3	.00003219	**3**	.00001350	**3**	.000003777	**3**	.000002012
4	.00002814	**4**	.00001013	**4**	.000002533	**4**	.000001759
5	.00002251	**5**	.000008105	**5**	.000002026	**5**	.000001407
6	.00001876	**6**	.000006754	**6**	.000001688	**6**	.000001172
7	.00001585	**7**	.000005789	**7**	.000001447	**7**	.0000009910
8	.00001407	**8**	.000005066	**8**	.000001266	**8**	.0000008795
9	.00001250	**9**	.000004503	**9**	.000001125	**9**	.0000007817
10	.00001125	**10**	.000004052	**10**	.000001013	**10**	.0000007036
11	.00001023	**11**	.000003684	**11**	.000000921	**11**	.0000006396
12	.000009380	**12**	.000003377	**12**	.000000844	**12**	.0000005863
13	.000008659	**13**	.000003117	**13**	.000000779	**13**	.0000005412
14	.000008040	**14**	.000002894	**14**	.000000723	**14**	.0000005025
15	.000007504	**15**	.000002701	**15**	.000000675	**15**	.0000004690
16	.000007035	**16**	.000002533	**16**	.000000633	**16**	.0000004397
17	.000006620	**17**	.000002384	**17**	.000000596	**17**	.0000004138
18	.000006252	**18**	.000002251	**18**	.000000562	**18**	.0000003908
19	.000005929	**19**	.000002133	**19**	.000000533	**19**	.0000003700
20	.000005628	**20**	.000002026	**20**	.000000506	**20**	.0000003518
21	.000005360	**21**	.000001929	**21**	.000000482	**21**	.0000003350
22	.000005116	**22**	.000001842	**22**	.000000460	**22**	.0000003198
23	.000004894	**23**	.000001762	**23**	.000000440	**23**	.0000003059
24	.000004689	**24**	.000001688	**24**	.000000422	**24**	.0000002931
25	.000004496	**25**	.000001621	**25**	.000000405	**25**	.0000002810
26	.000004329	**26**	.000001558	**26**	.000000389	**26**	.0000002706
27	.000004172	**27**	.000001501	**27**	.000000375	**27**	.0000002608
28	.000004019	**28**	.000001447	**28**	.000000361	**28**	.0000002512
29	.000003881	**29**	.000001397	**29**	.000000349	**29**	.0000002426
30	.000003752	**30**	.000001350	**30**	.000000337	**30**	.0000002345
31	.000003630	**31**	.000001307	**31**	.000000326	**31**	.0000002269
32	.000003516	**32**	.000001266	**32**	.000000316	**32**	.0000002198
33	.000003411	**33**	.000001223	**33**	.000000305	**33**	.0000002132
34	.000003310	**34**	.000001192	**34**	.000000298	**34**	.0000002069
35	.000003216	**35**	.000001157	**35**	.000000289	**35**	.0000002010
36	.000003128	**36**	.000001125	**36**	.000000281	**36**	.0000001955
37	.000003040	**37**	.000001095	**37**	.000000273	**37**	.0000001901
38	.000002961	**38**	.000001066	**38**	.000000266	**38**	.0000001851
39	.000002886	**39**	.000001039	**39**	.000000259	**39**	.0000001804
40	.000002814	**40**	.000001013	**40**	.000000253	**40**	.0000001759
41	.000002745	**41**	.0000009885	**41**	.0000002471	**41**	.0000001716
42	.000002680	**42**	.0000009649	**42**	.0000002412	**42**	.0000001675
43	.000002617	**43**	.0000009425	**43**	.0000002356	**43**	.0000001636
44	.000002558	**44**	.0000009211	**44**	.0000002303	**44**	.0000001599
45	.000002500	**45**	.0000009006	**45**	.0000002251	**45**	.0000001563
46	.000002446	**46**	.0000008810	**46**	.0000002202	**46**	.0000001529
47	.000002395	**47**	.0000008613	**47**	.0000002155	**47**	.0000001497
48	.000002344	**48**	.0000008443	**48**	.0000002110	**48**	.0000001465
49	.000002296	**49**	.0000008271	**49**	.0000002067	**49**	.0000001435
50	.000002251	**50**	.0000008105	**50**	.0000002026	**50**	.0000001407

Number	Log'thm of the Number	Recip'cal of the Number	Log'thm of the Recip'cal
$\pi =$ **3.14159265**	0.49715	0.3123098	$\bar{1}$.495572
$2\pi =$ 6.28318530	0.79818	0.159155	$\bar{1}$.201806
$3\pi =$ 9.42477795	0.97247	0.10603	$\bar{1}$.025428
$4\pi =$ 12.56637060	1.09921	0.079577	$\bar{2}$.900787
$5\pi =$ 15.70796325	1.19612	0.063662	$\bar{2}$.803880
$6\pi =$ 18.84955590	1.27530		
$7\pi =$ 21.99114855	1.34225		
$8\pi =$ 25.13274120	1.40024		
$9\pi =$ 28.27433385	1.45139		
$10\pi =$ 31.4159265	1.49715		
$11\pi =$ 34.55751915	1.53853		
$12\pi =$ 37.69911183	1.57633		
$13\pi =$ 40.84070448	1.61108		
$14\pi =$ 43.98229713	1.64327		
$15\pi =$ 47.12388978	1.68107		
$16\pi =$ 50.26548243	1.70126		
$17\pi =$ 53.40707508	1.72759		
$18\pi =$ 56.54866773	1.75241		
$19\pi =$ 59.69026038	1.77526		
$20\pi =$ 62.83185303	1.79817		
$\frac{2}{3}\pi =$ 2.0943950	0.32103		
$\pi^2 =$ **9.869604**	0.99430	0.101321	$\bar{1}$.005695
$\pi^3 =$ 31.006277	1.49145	0.0322515	$\bar{2}$.508543
$\pi^4 =$ 97.409091	1.98860	0.0102659	$\bar{2}$.011359
$\pi^5 =$ 306.01969	2.48575		
$\pi^6 =$ 961.38919	2.98290		
$\frac{\pi^2}{4} =$ 2.4674	0.392239		
$2\pi^2 =$ 19.739208	1.29512		
$\sqrt{\frac{\pi}{2}} =$ 1.253314	0.09806		
$4\pi^2 =$ 39.478416	1.59635		
$\sqrt{\pi} =$ **1.7724539**	0.24858	0.5641895	0.751417
$\sqrt[3]{\pi} =$ 1.4645919	0.16572		
$\sqrt[4]{\pi} =$ 1.331303	0.12388		
$\sqrt[6]{\pi} =$ 1.2102032	0.08286		
$\frac{1}{\pi} \times \sqrt[3]{\frac{1}{\pi}} =$ 0.68278	1.83428		
$\sqrt[3]{\frac{6}{\pi}} =$ 1.24070	0.09366		

Number	Log'thm of the Number	Recip'cal of the Number	Log'th of th Recip'
$\frac{\pi}{2} =$ 1.5707963	0.19612	0.6366	$\bar{1}$.80
$\frac{\pi}{3} =$ 1.0471975	0.02003	1.2732402	0.104
$\frac{\pi}{4} =$ 0.7853981	$\bar{1}$.89509		
$\frac{\pi}{5} =$ 0.6283185	$\bar{1}$.79818		
$\frac{\pi}{6} =$ 0.5235987	$\bar{1}$.71900		
$\frac{\pi}{7} =$ 0.448799	$\bar{1}$.65205		
$\frac{\pi}{8} =$ 0.3926990	$\bar{1}$.59406		
$\frac{\pi}{9} =$ 0.3490655	$\bar{1}$.54291		
$\frac{\pi}{10} =$ 0.3141592	$\bar{1}$.49715		
$\frac{\pi}{11} =$ 0.2855993	$\bar{1}$.45574		
$\frac{\pi}{12} =$ 0.2617993	$\bar{1}$.41797		
$\frac{\pi}{13} =$ 0.2416609	$\bar{1}$.383204		
$\frac{\pi}{14} =$ 0.2243994	$\bar{1}$.35100		
$\frac{\pi}{15} =$ 0.2094395	$\bar{1}$.32103		
$\frac{\pi}{16} =$ 0.1963495	$\bar{1}$.29303		
$\frac{\pi}{17} =$ 0.1847995	$\bar{1}$.26667		
$\frac{\pi}{18} =$ 0.1745329	$\bar{1}$.24187		
$\frac{\pi}{19} =$ 0.1653469	$\bar{1}$.21837		
$\frac{\pi}{20} =$ 0.1570796	$\bar{1}$.19609		
$\frac{\pi}{32} =$ 0.0981745	$\bar{2}$.99200		
$\frac{\pi}{64} =$ 0.0490872	$\bar{2}$.69097		
$\frac{\pi}{108} =$ 0.029088	$\bar{2}$.46373		
$\frac{\pi}{180} =$ 0.0174532	$\bar{2}$.24188		
$\sqrt{1} =$ 1.000000	0.0000000		
$\sqrt{2} =$ **1.41421**	0.1505108	0.707108	0.8
$\sqrt{3} =$ **1.73205**	0.2385479	0.578035	0.7
$\sqrt{4} =$ 2.00000	0.3010300	0.50000	0.6
$\sqrt{5} =$ 2.23607	0.3494718	0.44843	0.6
$\sqrt{6} =$ 2.44949	0.3890597	0.4032	0.6
$\sqrt{7} =$ 2.64575	0.4225406	0.3787	0.5
$\sqrt{10} =$ 3.16228	0.4999893		
$\sqrt{11} =$ 3.3166	0.5206931		
$\sqrt{12} =$ 3.4661	0.5398411		
$\sqrt{13} =$ 3.6055	0.5569655		
$\sqrt{14} =$ 3.7416	0.5730574		
$\sqrt{15} =$ 3.8729	0.5880363		
$\sqrt{17} =$ 4.1231	0.6152344		
$\sqrt{18} =$ 4.2426	0.6276321		
$\sqrt{19} =$ 4.3588	0.6393669		
$\sqrt{20} =$ 4.4721	0.6505115		

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Examples in Alternating-Currents

Vol. II.

A continuation of the subject matter contained in Volume I. is expected to be on sale early in the Fall of 1916.

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How to Make a Transformer

FOR LOW PRESSURES

Review in **"Machinery"**, May, 1915.

How to Make a Low-Pressure Transformer. By F. E. Austin. 14 pages, 4½ by 7 inches. Illustrated. Published by Prof. F. E. Austin, Hanover, N. H. Price 25 cents.

This pamphlet describes the construction of a transformer to reduce electrical pressure from 110 volts to about 8 volts as a minimum, for experimental purposes, such as operating low voltage tungsten lamps, ringing door bells, etc. The directions should enable an amateur to make a successful transformer having good efficiency. The author states that the efficiency of transformers made in accordance with his instructions has been found to be over 90 percent in many cases, and never below 85 per cent.

Review in **"Electrical World"** July 17, 1915.

How to Make a Transformer for Low Pressures. By Prof. F. E. Austin, Hanover, N. H. 14 pages, illus. Price 25 cents.

This booklet should prove useful to amateurs wishing to construct small transformers. The directions are clear, and satisfactory results should be obtainable by any reader who will carefully follow the instructions.

From **"Electrical Engineering"**, July, 1915.

How to Make a Transformer for Low Pressures is the title of a small book designed for "young America" by Professor F. E. Austin, Hanover, N. H. The book gives specific directions for procedure in constructing transformers, and the many transformers built according to the specifications have shown wonderfully high efficiencies for small devices. The small transformers, when connected with the ordinary alternating current house circuits, may be used to operate small electric lights, doorbells, small arc lights and direct current toy railways, operating five or six loaded trains at one time. The test is sufficiently suggestive throughout to invite initiative on the part of the reader to deviate from the 19 clauses of the specifications and effect many variations. The price of the book is 25 cents.

The above book may be obtained from the following:

"The Wireless Age", 450 Fourth Ave., New York, N. Y.

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Prof. F. E. Austin, Hanover, N. H.

Directions for Designing, Making and Operating High - Pressure Transformers.

Review from **"Electrical Engineering"**, July, 1915.

A new book entitled "DIRECTIONS FOR DESIGNING, MAKING AND OPERATING HIGH PRESSURE TRANSFORMERS," written for those experimenters who desire to construct their own apparatus, is also published by Prof. F. E. Austin. The book is a companion volume of "How to Make a Transformer for Low Pressures," but containing more working directions and useful talks such as loss due to Hysteresis, per cubic inch of iron core for various flux densities and frequencies; and data applying to copper magnet wire. The book is well illustrated with half-tone and line cuts showing special methods of procedure, fundamental theories, and finished apparatus. It is written in simple English, is full of technical information and new ideas relating to methods of design and construction, and will prove of great assistance to those who are pursuing correspondence courses or regular college courses. The price of the book is 50 cents.

Review from **"Electrical World"**, July 17, 1915.

Directions for Designing, Making and Operating High-Pressure Transformers. By Prof. F. E. Austin, Hanover, N. H. 46 pages, 21 illus. Price 50 cents.

As a complete problem or example in transformer design this little book has appreciable educational value. It gives clear directions and provides the necessary data for constructing a practical 3-kw., 20,000-volt transformer. Notwithstanding the fact that theoretical considerations and explanations are not omitted, this book alone would not give the necessary training in fundamental principles to the engineering student. The book may be read and applied to useful ends without requiring a concentrated and sustained mental effort but in conjunction with other class books, both this and the booklet by the same author on low-tension transformer design should be of value.

Review from **"Machinery"**, May, 1915.

Directions for Designing, Making and Operating High-Pressure Transformers. By F. E. Austin. 46 pages, 4¾ by 7½ inches. 21 illustrations. Published by Prof. F. E. Austin, Hanover, N. H. Price 50 cents.

This pamphlet is a companion to the author's work on making a transformer for low pressures. It describes the making of a step-up transformer, giving 20,000 volts, for wireless telegraphs, telephones, for operating tube lamps and X-ray tubes. The mathematical matter is treated in a simple way that is well within the comprehension of the amateur who would be interested in building a high-pressure transformer for experimental purposes. All materials are specified and all calculations are worked out for the model described. Probably no exercise would give the average student a firmer grasp of electrical principles than the building of a piece of apparatus like this transformer.

The book dealing with high-pressure transformers may be obtained from the following:

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